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First-Fit coloring of bounded tolerance graphs

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ABSTRACT

Let $G = (V, E)$ be a graph. A tolerance representation of G is a set $\mathcal{I} = \{I_v : v \in V\}$ of intervals and a set $t = \{t_v : v \in V\}$ of nonnegative reals such that $xy \in E$ iff $I_x \cap I_y \neq \emptyset$ and $\|I_x \cap I_y\| \geq \min\{t_x, t_y\}$; in this case G is a tolerance graph. We refine this definition by saying that G is a p-tolerance graph if $t_v/|I_v| \leq p$ for all $v \in V$.

A Grundy coloring g of G is a proper coloring of V with positive integers such that for every positive integer i, if $i < g(v)$ then v has a neighbor u with $g(u) = i$. The Grundy number $\Gamma(G)$ of G is the maximum integer k such that G has a Grundy coloring using k colors. It is also called the First-Fit chromatic number.

For fixed $0 \leq p < 1$ we prove that if G is a p-tolerance graph then, $\Gamma(G) = \Theta\left(\frac{\omega(G)}{1-p}\right)$, and in particular, $\Gamma(G) \leq 8\left\lceil\frac{1}{1-p}\right\rceil\omega(G)$. Also, we show how restricting p forbids induced copies of $K_{s,s}$. Finally, we observe that there exist 1-tolerance graphs G with $\omega(G) = 2$ and arbitrarily large Grundy number.

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1. Introduction

In this paper we study the performance of the online coloring algorithm First-Fit on tolerance graphs. First, we review some notation and definitions. For an interval $I = [a, b]$, let $\|I\|$ denote the length $b - a$ of I. For a graph G, denote its clique number by $\omega(G)$, its independence number by $\alpha(G)$ and its chromatic number by $\chi(G)$. We say that a graph class $\mathcal{G}$ is χ-bounded if there exists a function f such that $\chi(G) \leq f(\omega(G))$ for all $G \in \mathcal{G}$. In this case f is called a *bounding function* for $\mathcal{G}$. For a graph H let $\mathrm{Forb}(H)$ denote the class of graphs that do not contain H as an induced subgraph.

1.1. Online and First-Fit coloring and Grundy numbers

An *online graph* $G^{\prec}$ is a graph G together with an ordering $\prec$ of its vertices. This ordering is called the *presentation* of G. Let $G_i^{\prec}$ denote the online graph induced by the first i vertices of $\prec$. An *online coloring algorithm* is an algorithm $\mathcal{A}$ that colors the vertices of G so that the color of the ith vertex v_i depends only on $G_i^{\prec}$. The number of colors used by $\mathcal{A}$ on $G^{\prec}$ is denoted by $\chi_{\mathcal{A}}(G^{\prec})$ and $\chi_{\mathcal{A}}(G)$ is the maximum of $\chi_{\mathcal{A}}(G^{\prec})$ over all possible orderings $\prec$. A class $\mathcal{G}$ of graphs is online χ-bounded if there exist an online algorithm $\mathcal{A}$ and a function f such that $\chi_{\mathcal{A}}(G^{\prec}) \leq f(\omega(G))$ for all graphs $G \in \mathcal{G}$ and all presentations $\prec$ of G. In this case f is called an online *bounding function* for $\mathcal{G}$.

First-Fit (*FF*) is the online algorithm that colors the ith vertex v_i of an online graph $G^{\prec}$ with the least positive integer that has not been used to color any of its neighbors in $G_i^{\prec}$. A class $\mathcal{G}$ is First-Fit χ-bounded if there exists a function f such that

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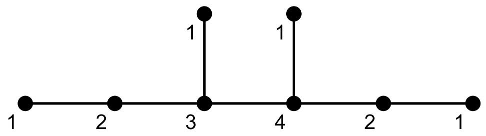

Fig. 1. Graph with $\Gamma(G) = 4$.

$\chi_{FF}(G^{\prec}) \le f(\omega(G))$ for all graphs $G \in \mathcal{G}$ and all presentations $\prec$ of G. In this case f is called a First-Fit *bounding function* for $\mathcal{G}$. Finally, $\chi_{FF}(\mathcal{G}) = \max_{G\in\mathcal{G}} \chi_{FF}(G)$.

A *Grundy coloring* of a graph $G = (V, E)$ is a proper coloring g of G with positive integers such that

$$\forall v \in V\ \forall k \in \mathbb{Z}^+ \quad (k < g(v) \Rightarrow \exists u \in N(v)\, g(u) = k). \tag{$*$}$$

The *Grundy number* $\Gamma(G)$ of G is the maximum integer k such that G has a Grundy coloring using the color k. An example of a Grundy coloring is shown above in Fig. 1. It is easy to see that $\Gamma(G) = \chi_{FF}(G)$: First-Fit produces a Grundy coloring, and every Grundy coloring can be realized by First-Fit, if the vertices are presented so that $g(x) < g(y)$ implies that $x \prec y$. In proofs it is more convenient to consider Grundy number than First-Fit colorings, since then we can ignore presentations. We also define a *weak Grundy coloring* to be a possibly improper coloring that satisfies $(*)$.

1.2. Interval and tolerance graphs

A graph $G = (V, E)$ is an *interval graph* if for each vertex $x \in V$ there exists a closed interval $I_x = [L(x), R(x)]$ of **R** such that $xy \in E$ if and only if $I_x \cap I_y \neq \emptyset$. In this case the set $\mathcal{I} := \{I_v : v \in V\}$ is called an *interval representation* of G. If $R(x) < L(y)$ we write $I_x < I_y$. Then $\mathcal{I}$ is also an interval representation of the *interval order* defined on V by $x < y$ iff $I_x < I_y$ and G is the cocomparability graph of this order.

Tolerance graphs were introduced by Golumbic and Monma [7] as a natural generalization of interval graphs. A graph $G = (V, E)$ is a *tolerance graph* if for each vertex $x \in V$ there exists a closed interval $I_x = [L(x), R(x)]$ of **R** and a nonnegative real t_x such that $xy \in E$ if and only if $I_x \cap I_y \neq \emptyset$ and $\|I_x \cap I_y\| \ge \min\{t_x, t_y\}$. In this case $\langle \mathcal{I}, t\rangle$ is called a *tolerance representation* of G, where t maps $x \mapsto t_x$. It is useful for us to introduce the following classification of tolerance graphs. Define G to be a *p-tolerance graph* if it has a tolerance representation $\langle \mathcal{I}, t\rangle$ such that $t_x / \|I_x\| \le p$ for all $x \in V$. Then interval graphs are 0-tolerance graphs. In the past, 1-tolerance graphs have been extensively studied under the name *bounded tolerance graphs* and $\frac{1}{2}$-tolerance graphs have been studied under the name *totally bounded tolerance graphs*. If one views tolerance graphs as imprecise interval graphs, then p measures the degree of imprecision.

Let $\mathcal{T}_p$ denote the class of p-tolerance graphs and $\mathcal{T}_{p,w}$ be the restriction of $\mathcal{T}_p$ to graphs with clique size at most w. Golumbic and Monma [7] proved that bounded tolerance graphs are cocomparability graphs and Golumbic et al. [8] proved that all tolerance graphs are perfect. For further details the reader is referred to the excellent books [6] by Golumbic on algorithmic graph theory and [9] by Golumbic and Trenk on tolerance graphs.

1.3. Old and new results

There has been extensive research on the online coloring of interval graphs. Kierstead and Trotter [19] showed that there exists an online algorithm $\mathcal{A}$ such that $\chi_{\mathcal{A}}(G^{\prec}) \le 3\omega(G) - 2$ for any online interval graph $G^{\prec}$ and that no online algorithm can do better on the class of all interval graphs. Kierstead proved that every online interval graph $G^{\prec}$ satisfies $\chi_{FF}(G^{\prec}) \le 40\omega(G)$. This upper bound was improved to $26\omega(G)$ in [17], before Pemmaraju et al. [22] introduced a beautiful new technique to reduce the bound to $10\omega(G)$. Brightwell et al. [3], and later more elegantly, Narayanaswamy and Subhash Babu [21], used easy modifications of this technique to get $8\omega(G)$. Chrobak and Ślusarek [4] proved that any First-Fit bounding function f for the class of interval graphs satisfies $f(k) \ge 4.4k - b$ for some constant b. Kierstead et al. [20] have recently improved this to: for all $\varepsilon > 0$ there exists b such that for all k, $f(k) \ge (5 - \varepsilon)k - b$.

Gyárfás [10], and independently Sumner [24], conjectured that Forb(T) is χ-bounded for every tree T. Gyárfás et al. [11] proved this for the special case that T has radius 2 and $\omega = 2$. Kierstead and Penrice [16] proved the general result for radius 2 trees. Kierstead et al. [18] showed that Forb(T) is online χ-bounded for radius 2 trees T. In particular, Forb$(SK_{1,3})$ is online χ-bounded, where $SK_{1,3}$ is the radius 2 tree obtained by subdividing each edge of $K_{1,3}$. Since every bounded tolerance graph is a cocomparability graph, and no cocomparability graph induces $SK_{1,3}$, the class of bounded tolerance graphs is online χ-bounded. However, the known bounding function is superexponential. It is an open question whether all tolerance graphs are cocomparability graphs. If true, then the class of tolerance graphs is online χ-bounded.

In [13], Kierstead showed that cocomparability graphs are not First-Fit χ-bounded by constructing posets with width 2, whose cocomparability graphs have arbitrarily large Grundy number (and clique number 2). Hiraguchi [12] showed that the dimension of a poset is at most its width, and so these posets are two dimensional. Hence their cocomparability graphs are permutation graphs. Golumbic and Monma [7] showed that permutation graphs are bounded tolerance graphs. In fact, they can be represented by intervals so that $t_x = \|I_x\|$ for all vertices x. Such a tolerance representation for a bounded tolerance graph with Grundy number 7 is shown in Fig. 2. In the figure, the color of an interval (more precisely, of the vertex

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