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The external constraint 4 nonempty part sandwich problem☆

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ABSTRACT

List partitions generalize list colourings. Sandwich problems generalize recognition problems. The polynomial dichotomy (NP-complete versus polynomial) of list partition problems is solved for 4-dimensional partitions with the exception of one problem (the LIST STUBBORN PROBLEM) for which the complexity is known to be quasipolynomial. Every partition problem for 4 nonempty parts and only external constraints is known to be polynomial with the exception of one problem (the $2K_2$-PARTITION PROBLEM) for which the complexity of the corresponding list problem is known to be NP-complete. The present paper considers external constraint 4 nonempty part sandwich problems. We extend the tools developed for polynomial solutions of recognition problems obtaining polynomial solutions for most corresponding sandwich versions. We extend the tools developed for NP-complete reductions of sandwich partition problems obtaining the classification into NP-complete for some external constraint 4 nonempty part sandwich problems. On the other hand and additionally, we propose a general strategy for defining polynomial reductions from the $2K_2$-PARTITION PROBLEM to several external constraint 4 nonempty part sandwich problems, defining a class of $2K_2$-hard problems. Finally, we discuss the complexity of the Skew Partition Sandwich Problem.

1. Introduction

The polynomial dichotomy of a class of problems distinguishes the complexity into polynomial time and NP-complete and has been much studied for the class of list partition problems, a generalization of list colourings [16]. For list M-partition problems, where a symmetric matrix M defines a list partition problem, the dichotomy into quasipolynomial and NP-complete was completely determined, when M is a 4-by-4 matrix [16]. Subsequently, the polynomial dichotomy was determined with the exception of just one 4-by-4 matrix, for which the so-called LIST STUBBORN PROBLEM remains classified as quasipolynomial [1].

Different choices of the matrix M lead to many well-known graph theoretic problems including the problem of recognizing split graphs and their generalizations, the (k, l)-graphs; of finding homogeneous sets; of finding structured cutsets, for instance clique, stable, star or skew. These combinatorial graph theoretic problems seek a partition of the vertices of a given graph into subsets satisfying certain constraints *internally* (a set may be required to be stable, complete) and *externally* (two sets may be required to be completely adjacent, completely nonadjacent). Additional constraints with respect to be nonempty parts, or having at least a fixed number of elements are captured by the addition of *lists* to the input [16].

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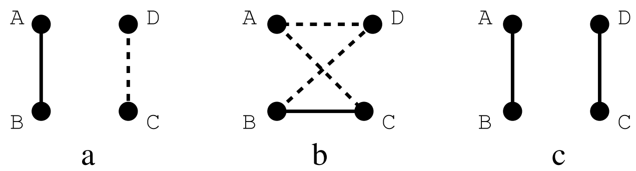

Fig. 1. (a) Skew partition, (b) 1-join composition, and (c) $2K_2$-partition.

Consider the three partition problems represented in Fig. 1. Each problem seeks for a partition of the vertex set of the input graph into 4 nonempty parts according to *external* constraints represented by a *model* graph. Throughout the paper, we shall employ the following representation for external constraint partitions into 4 parts: we represent each of the 4 nonempty parts by a labelled vertex A, B, C, and D—we keep the same labels throughout the text; the external constraint that there are all edges between two parts by a continuous line; and the external constraint that there are no edges between two parts by a dotted line. Special interest in the context of decomposing perfect graphs has been given to *skew partition* and to 1-*join composition* [3,5,4,7]. Skew partition was defined as a generalization of clique cutsets and star cutsets [5] and subsequently the LIST SKEW PARTITION PROBLEM was proved to be polynomial [12]. More recently, a fast polynomial-time algorithm for SKEW PARTITION PROBLEM was presented [24].

All partition problems into 4 nonempty parts according to external constraints are classified as polynomial with the exception of just one such problem, the $2K_2$-PARTITION PROBLEM [10]. Remark that the LIST $2K_2$-PARTITION PROBLEM is classified as NP-complete [16].

In their seminal paper, Golumbic et al. [20] have introduced sandwich problems and studied this class of problems with respect to several subclasses of perfect graphs proving that GRAPH SANDWICH PROBLEM FOR SPLIT GRAPHS remains polynomial. On the other hand, they proved that GRAPH SANDWICH PROBLEM FOR PERMUTATION GRAPHS turns out to be NP-complete. Since then, Professor Martin Charles Golumbic has contributed with many results to the study of this class of problems [20,19,21]. Graph sandwich problems [20,19,21,23,22] are generalized recognition problems arising from applications in computational biology. The polynomial dichotomy for (k, l)-graph sandwich problems is completely determined [8]. The polynomial dichotomy for 3 nonempty part sandwich problems is completely determined [27,25]. The present work considers the polynomial dichotomy for the class of external constraint 4 nonempty part sandwich problems.

Since a sandwich problem generalizes a recognition problem, the interest is to search for problems for which the recognition is polynomial but the sandwich version is NP-complete. For instance, the 1-JOIN COMPOSITION RECOGNITION PROBLEM is polynomial whereas the 1-JOIN COMPOSITION SANDWICH PROBLEM turns out to be NP-complete [13]. Surprisingly, both STAR CUTSET RECOGNITION PROBLEM and CLIQUE CUTSET RECOGNITION PROBLEM are polynomial, whereas only CLIQUE CUTSET SANDWICH PROBLEM turns out to be NP-complete [27]. A further interesting example that separates the complexity of problems that are equally classified as polynomial regarding recognition occurs in the polynomial dichotomy for the class of 3 nonempty part sandwich problems: the addition of distinct internal constraints separates versions of HOMOGENEOUS SET SANDWICH PROBLEM into polynomial or NP-complete [2,25].

On the other hand, we remark that polynomial graph sandwich problems are rare. In the original paper by Golumbic et al. [20], where this generalization of recognition problems was proposed [20], several classes of perfect graphs were considered with NP-completeness proofs for the corresponding sandwich problems with only two exceptions: split graphs and cographs, for which the sandwich problems are proved to be polynomial. Further work found additional two examples of polynomial graph sandwich problems also arising in the context of perfect graphs: HOMOGENEOUS SET SANDWICH PROBLEM [2], and STAR CUTSET SANDWICH PROBLEM [27]. More recently, by considering 3 nonempty part sandwich problems, additional polynomial graph sandwich problems were found [25].

In the present work, our goal is twofold. First, to study the complexity of $2K_2$-PARTITION RECOGNITION PROBLEM, the only external constraint 4 nonempty part recognition problem for which the complexity remains open [6,10]. Note that so far all classified external constraint 4 nonempty part recognition problems are polynomial [10,16,12]. Second, to establish the polynomial dichotomy for the class of external constraint 4 nonempty part sandwich problems.

The present work contributes by reducing the $2K_2$-PARTITION RECOGNITION PROBLEM to several external constraint 4 nonempty part sandwich problems, which proposes the study of the class of $2K_2$-hard problems; by extending tools developed for NP-complete reductions of sandwich partition problems obtaining the classification into NP-complete for some external constraint 4 nonempty part sandwich problems; and by generalizing tools previously used in recognition problems to sandwich problems which enables the classification of most external constraint 4 nonempty part sandwich problems as polynomial. The difficulty of the SKEW PARTITION SANDWICH PROBLEM is considered by proving that the version with two fixed vertices in parts A or B is NP-complete.

2. Definitions

Consider an undirected, finite, simple graph $G = (V(G), E(G))$ and the problem of finding a partition of $V(G)$ into nonempty subsets satisfying *internal* or *external* constraints. An internal constraint refers to constraints within the parts as to be a clique, or an independent set. An external constraint refers to constraints between different parts as to be completely