Available online at www.sciencedirect.com
ScienceDirect
Discrete Applied Mathematics 155 (2007) 82 – 91
DISCRETE APPLIED MATHEMATICS
www.elsevier.com/locate/dam

Packing directed cycles efficiently☆

Zeev Nutov[a], Raphael Yuster[b]

[a]Department of Computer Science, The Open University of Israel, Tel Aviv, Israel
[b]Department of Mathematics, University of Haifa, Haifa 31905, Israel

Received 26 September 2004; received in revised form 15 May 2005; accepted 20 April 2006
Available online 22 June 2006

Abstract

Let G be a simple digraph. The *dicycle packing number* of G, denoted $v_c(G)$, is the maximum size of a set of arc-disjoint directed cycles in G. Let G be a digraph with a nonnegative arc-weight function w. A function ψ from the set $\mathscr{C}$ of directed cycles in G to R_+ is a *fractional dicycle packing* of G if $\sum_{e \in C \in \mathscr{C}} \psi(C) \leqslant w(e)$ for each $e \in E(G)$. The *fractional dicycle packing number*, denoted $v_c^*(G, w)$, is the maximum value of $\sum_{C \in \mathscr{C}} \psi(C)$ taken over all fractional dicycle packings ψ. In case $w \equiv 1$ we denote the latter parameter by $v_c^*(G)$.

Our main result is that $v_c^*(G) - v_c(G) = o(n^2)$ where $n = |V(G)|$. Our proof is algorithmic and generates a set of arc-disjoint directed cycles whose size is at least $v_c(G) - o(n^2)$ in randomized polynomial time. Since computing $v_c(G)$ is an NP-Hard problem, and since almost all digraphs have $v_c(G) = \Theta(n^2)$ our result is a FPTAS for computing $v_c(G)$ for almost all digraphs.

The result uses as its main lemma a much more general result. Let $\mathscr{F}$ be any fixed family of oriented graphs. For an oriented graph G, let $v_{\mathscr{F}}(G)$ denote the maximum number of arc-disjoint copies of elements of $\mathscr{F}$ that can be found in G, and let $v_{\mathscr{F}}^*(G)$ denote the fractional relaxation. Then, $v_{\mathscr{F}}^*(G) - v_{\mathscr{F}}(G) = o(n^2)$. This lemma uses the recently discovered *directed regularity lemma* as its main tool.

It is well known that $v_c^*(G, w)$ can be computed in polynomial time by considering the dual problem. We present a polynomial algorithm that finds an optimal fractional dicycle packing. Our algorithm consists of a solution to a simple linear program and some minor modifications, and avoids using the ellipsoid method. In fact, the algorithm shows that a maximum fractional dicycle packing with at most $O(n^2)$ dicycles receiving nonzero weight can be found in polynomial time.

Keywords: Cycles; Packing; Digraph

1. Introduction

All graphs and digraphs considered here are finite and have no loops, parallel arcs or isolated vertices. For the standard terminology used the reader is referred to [5]. We use the terms *digraph* and *dicycle* to refer to a directed graph and a directed cycle, respectively.

We consider the following fundamental problem in algorithmic graph-theory. Given a digraph G, how many arc-disjoint cycles can be packed into G? Define the *dicycle packing number* of G, denoted $v_c(G)$, to be the maximum size of a set of arc-disjoint dicycles in G. We also consider the fractional relaxation of this problem. Let R_+ denote

☆ A preliminary version of this paper appeared in Proceedings of the 29th International Symposium on Mathematical Foundations of Computer Science, Prague, Czech Republic, 2004.

E-mail addresses: nutov@openu.ac.il (Z. Nutov), raphy@research.haifa.ac.il (R. Yuster).

0166-218X/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2006.04.033

the set of nonnegative reals. A *fractional dicycle packing* of G is a function ψ from the set $\mathscr{C}$ of dicycles in G to R_+, satisfying $\sum_{e\in C\in\mathscr{C}} \psi(C) \leqslant 1$ for each $e \in E(G)$. Letting $|\psi| = \sum_{C\in\mathscr{C}}\psi(C)$, the *fractional dicycle packing number*, denoted $v_c^*(G)$, is defined to be the maximum of $|\psi|$ taken over all fractional dicycle packings ψ. Since a dicycle packing is also a fractional dicycle packing, we always have $v_c^*(G) \geqslant v_c(G)$. The notion of a fractional dicycle packing can be extended to digraphs with nonnegative arc weights. In this case, we require that $\sum_{e\in C\in\mathscr{C}} \psi(C) \leqslant w(e)$ for each $e \in E(G)$ where $w(e)$ is the weight of e. We denote by $v_c^*(G, w)$ the corresponding *fractional dicycle packing number* where $w : E \to R_+$ is the weight function.

Problems concerning packing arc-disjoint or vertex-disjoint dicycles in digraphs have been studied extensively (see, e.g., [4,15]). It is well known that computing $v_c(G)$ (and hence finding a maximum dicycle packing) is an NP-Hard problem. Even the very special case of deciding whether a digraph has a triangle decomposition is known to be NP-Complete (see, e.g., [7] for a more general theorem on the NP-Completeness of such decomposition problems). Currently, the best approximation algorithm for this problem [13] has an approximation ratio of $O(n^{1/2})$ which is also an upper bound for the integrality gap. Thus, it is interesting to find out when $v_c(G)$ and $v_c^*(G)$ are "close" as this immediately yields an efficient approximation algorithm for this NP-Hard problem. Our main result shows that the two parameters differ by at most $o(n^2)$, thus giving an approximation algorithm with an $o(n^2)$ additive error term.

Theorem 1.1. *If G is an n-vertex digraph then* $v_c^*(G) - v_c(G) = o(n^2)$ *and a set of at least* $v_c(G) - o(n^2)$ *arc-disjoint dicycles can be generated in randomized polynomial time. There are n-vertex graphs G for which* $v_c^*(G) - v_c(G) = \Omega(n^{3/2})$.

The $o(n^2)$ additive error term is only interesting if the graph G is dense and $v_c(G) = \Theta(n^2)$. This, however, is the case for almost all digraphs, as it is known (and easy) that the directed random graph $G(n, p)$ has $v_c(G) = \Theta(n^2)$ for any constant p, $0 < p < 1$ (in this model each of the $n(n-1)$ arcs has probability p of being selected). There are also many other explicit constructions of digraphs with $v_c(G) = \Theta(n^2)$ which do not resemble a typical element of $G(n, p)$. The second part of Theorem 1.1 shows that the $o(n^2)$ error term in Theorem 1.1 *cannot* be replaced with $o(n^{1.5})$.

The first part of Theorem 1.1 uses as its main lemma a much more general result concerning packings of oriented graphs. Recall that an oriented graph is a directed graph without 2-cycles. Let $\mathscr{F}$ be any given (finite or infinite) family of oriented graphs. For an oriented graph G, let $v_{\mathscr{F}}(G)$ denote the maximum number of arc-disjoint copies of elements of $\mathscr{F}$ that can be found in G, and let $v_{\mathscr{F}}^*(G)$ denote the respective fractional relaxation. We prove the following.

Theorem 1.2. *For any given family of oriented graphs, if G is an n-vertex oriented graph then* $v_{\mathscr{F}}^*(G) - v_{\mathscr{F}}(G) = o(n^2)$. *Furthermore, a set of at least* $v_{\mathscr{F}}(G) - o(n^2)$ *arc-disjoint elements of* $\mathscr{F}$ *can be generated in randomized polynomial time.*

The first part of Theorem 1.1 is a consequence of Theorem 1.2 by considering the family $\mathscr{F}$ of all directed cycles of length at least 3. An initial preprocessing step allows us to get rid of the 2-cycles of G.

We note that an *undirected* version of Theorem 1.2 has been recently proved by the second author [18] extending an earlier result of Haxell and Rödl [11] dealing with single element families. The proof of Theorem 1.2 makes use of the *directed regularity lemma* that has been used implicitly in [6] and proved in [2], and that enables us to overcome several difficulties that do not occur in the undirected case.

It is well known that $v_c^*(G, w)$ can be computed in polynomial time by considering the dual problem whose solution is known to be computable in polynomial time [14]. This follows from the strong duality theorem. It also follows from the same method used in [12] that, using the ellipsoid method and a separation oracle which exists for the dual problem, an optimal fractional dicycle packing can also be generated in polynomial time. However, we present a much simpler algorithm which avoids using the ellipsoid method and merely consists of solving some related simple linear program and slightly modifying the solution. In particular, we prove the following result.

Theorem 1.3. *If G is an n-vertex digraph associated with a nonnegative arc-weight function w, then a maximum fractional dicycle packing can be computed in polynomial time. Furthermore, a maximum fractional dicycle packing with at most* $O(n^2)$ (*resp.* $O(n^3)$) *dicycles receiving nonzero weight can be found in* (*resp. strongly*) *polynomial time.*

In the next two sections we prove our results.