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Graphs of separability at most 2

Ferdinando Cicalese[a], Martin Milanič[b,*]

[a] Dipartimento di Informatica, University of Salerno, Fisciano, Italy

[b] University of Primorska, FAMNIT and PINT, Koper, Slovenia

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ABSTRACT

We introduce *graphs of separability at most k* as graphs in which every two non-adjacent vertices are separated by a set of at most k other vertices. Graphs of separability at most k arise in connection with the Parsimony Haplotyping problem from computational biology. For $k \in \{0, 1\}$, the only connected graphs of separability at most k are complete graphs and block graphs, respectively. For $k \geq 3$, graphs of separability at most k form a rich class of graphs containing all graphs of maximum degree k.

We prove several characterizations of graphs of separability at most 2, which generalize complete graphs, cycles and trees. The main result is that every connected graph of separability at most 2 can be constructed from complete graphs and cycles by pasting along vertices or edges, and vice versa, every graph constructed this way is of separability at most 2. The structure theorem has nice algorithmic implications—some of which cannot be extended to graphs of higher separability—however certain optimization problems remain intractable on graphs of separability 2. We then characterize graphs of separability at most 2 in terms of minimal forbidden induced subgraphs and minimal forbidden induced minors. Finally, we discuss the possibilities of extending these results to graphs of higher separability.

1. Introduction

Let $G = (V, E)$ be a graph. The *separability* $\text{sep}_G(x, y)$ of two distinct non-adjacent vertices x, y in G is defined as the minimum cardinality of a set $S \subseteq V$ such that x and y are in different components of $G - S$. We define the *separability of a graph G*, denoted by $\text{sep}(G)$, as the maximum over all separabilities of non-adjacent vertex pairs (unless G is complete, in which case we define its separability to be 0). Notice that by definition, graphs of separability at most k are precisely the graphs in which every two non-adjacent vertices can be separated by removing a set of at most k other vertices. Hence, by Menger's theorem, the separability of G is equal to the maximum number of internally vertex-disjoint paths connecting two non-adjacent vertices.

Graphs of separability at most k arise naturally in connection with the Parsimony Haplotyping problem from computational biology. (This connection is detailed in Section 6.) We are interested in characterizations and structural properties of graphs of separability at most k, for small values of k. It can be easily seen that for every k, the set $\mathcal{G}_k$ of graphs of separability at most k is closed under vertex deletions; hence, with every graph $G \in \mathcal{G}_k$, the class $\mathcal{G}_k$ contains all induced subgraphs of G. Such graph classes are called *hereditary*. This family of graph classes is of particular interest, since hereditary (and only hereditary) classes admit a uniform description in terms of forbidden induced subgraphs. For a set $\mathcal{F}$ of graphs, we say that a graph G is *$\mathcal{F}$-free* if it does not contain an induced subgraph isomorphic to a member of $\mathcal{F}$. Given a hereditary class $\mathcal{G}$, denote by $\mathcal{F}$ the set of all graphs G with the property that $G \notin \mathcal{G}$ but $H \in \mathcal{G}$ for every proper induced subgraph

* Corresponding author. Fax: +386 5 611 75 71.
E-mail addresses: cicalese@dia.unisa.it (F. Cicalese), martin.milanic@upr.si (M. Milanič).

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H of G. The set $\mathcal{F}$ is said to be the set of forbidden induced subgraphs for $\mathcal{G}$, and $\mathcal{G}$ is precisely the class of $\mathcal{F}$*-free* graphs. The set $\mathcal{F}$ can be either finite or infinite, and many interesting classes of graphs can be characterized as being $\mathcal{F}$-free for some family $\mathcal{F}$. Such characterizations can be useful for establishing inclusion relations among hereditary graph classes, and were obtained for, among others, even-signable graphs [13], universally signable graphs [14,17], and perfect graphs in the famous Strong Perfect Graph Theorem conjectured by Berge in 1961 [3] and proved by Chudnovsky et al. in 2006 [8].

There are also theorems that elucidate the structure of graphs in a certain hereditary class by showing that every graph in the class either belongs to one of a few basic classes (in which case it has a prescribed and relatively transparent structure) or has one of a set of prescribed structural faults, along which it can be decomposed in a useful way. Several such decomposition results were obtained in recent years, including those for Meyniel graphs [6], perfect graphs [8], cap-free graphs [12], universally signable graphs [14,17], even-hole-free graphs [15], graphs without odd holes, parachutes or proper wheels [11], odd-hole-free graphs [16] and (diamond, even-hole)-free graphs [29]. A few results of a stronger type are also known, in which the decomposition can also be reversed in the sense that a graph is in the class if and only if it can be constructed by gluing basic graphs along the decompositions prescribed. Such *composition* results are known for example for chordal graphs [23], claw-free graphs [9], graphs with no cycle with a unique chord [36] and bull-free graphs [7]. Decomposition results often have nice algorithmic consequences and provide means for obtaining bounds on certain graph parameters in terms of others.

We initiate the study of the structural properties of graphs of separability at most k, for small values of k. For $k \in \{0, 1\}$, graphs of separability at most k are completely understood: graphs of separability 0 are precisely the disjoint unions of complete graphs, and graphs of separability at most 1 are precisely the *block graphs*, that is, graphs every block of which is complete. From this description, a forbidden induced subgraph characterization is easy to obtain, and it is clear how to build such graphs from the complete graphs. For $k \geq 3$, graphs of separability at most k form a rich class of graphs containing all graphs of maximum degree k, as well as all pairwise k-separable graphs (defined by Miller [32]). The main focus of this paper is on the class of graphs of separability at most 2. These graphs form a common generalization of complete graphs, cycles and trees, and more generally, block graphs, cacti (graphs in which every edge belongs to at most one cycle), forests, and block-cactus graphs (graphs in which every block induces either a complete graph or a cycle).

Our results. We show in Section 2 that graphs of separability at most 2 are precisely the graphs that can be built from complete graphs and cycles by an iterative application of the disjoint union operation and of pasting two disjoint graphs along a vertex or along an edge. In Section 3 we examine the unboundedness of the tree-width and the clique-width, when restricted to graphs of separability at most 2. We show that the structure theorem leads to polynomial time solvability of several generally NP-hard problems, in this class. The structure theorem also implies the existence of an efficient recognition algorithm of graphs of separability at most 2. Interestingly, some well-known hard problems remain intractable when restricted to graphs of separability at most 2. In Section 4, we characterize the graphs of separability at most 2 in terms of minimal forbidden induced subgraphs and minimal forbidden induced minors; these characterizations imply that every graph of separability at most 2 is universally signable. In Section 5 we summarize the results for graphs of separability at most k, and in Section 6 we explain how graphs of separability at most k arise in connection with the Parsimony Haplotyping problem from computational biology. Section 7 concludes the paper with some open problems.

This paper is the full version of the conference paper [10]. It includes, in addition to all proofs, a separate section devoted to graphs of separability at most k and a description of the connection between graphs of separability at most k and the Parsimony Haplotyping problem.

Notation and definitions. All graphs considered are finite, simple and undirected. As usual, C_n and K_n denote the cycle and the complete graph on n vertices, respectively, and $K_{s,t}$ the complete bipartite graph with parts of size s and t. For a vertex $x \in V(G)$, we denote by $N(x)$ the neighborhood of x, i.e., the set of vertices adjacent to x. The *degree* of x is the size of its neighborhood. For a set $A \subseteq V(G)$, we denote by $N(A)$ the set $\cup_{a\in A}\{u \in N(a) : u \notin A\}$, and for sets $A, B \subseteq V(G)$ we define $N_B(A) := N(A) \cap B$. Unless stated otherwise, m and n will denote the number of edges and vertices of the graph under consideration. A graph G is *chordal* if every cycle in G on at least four vertices has a chord (an edge connecting two non-consecutive vertices of the cycle). A *clique* in a graph G is a set of pairwise adjacent vertices. A *separating clique* in a graph G is a clique C in G whose removal disconnects G. An *independent set* in a graph G is a set of pairwise non-adjacent vertices. A *cut-vertex* of a connected graph G is a vertex whose removal disconnects the graph. A 2*-connected graph* is a connected graph on at least three vertices and with no cut-vertices. A *2-connected component* of a graph G is a maximal subgraph of G that is 2-connected. A *block* of a connected graph G is either a 2-connected component of G or an edge whose removal separates the graph. We say that a graph G is obtained from two graphs G_1 and G_2 by *pasting along a k-clique*, and denote this by $G = G_1 \oplus_k G_2$, if for some $r \leq k$ there exist two r-cliques $K_1 = \{x_1, \ldots, x_r\} \subseteq V(G_1)$ and $K_2 = \{y_1, \ldots, y_r\} \subseteq V(G_2)$ such that G is isomorphic to the graph obtained from the disjoint union of G_1 and G_2 by identifying each x_i with y_i, for all $i = 1, \ldots, r$. In particular, if $k = 0$, then $G_1 \oplus_k G_2$ is the disjoint union of G_1 and G_2. For terms left undefined, we refer the reader to [22].

2. A structure theorem for graphs of separability at most 2

Complete graphs and cycles are graphs of separability at most 2. The main result of this section is the following theorem, showing that complete graphs and cycles form the main building blocks for every graph of separability at most 2.