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Equitable colorings of Kronecker products of graphs[☆]

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ABSTRACT

For a positive integer k, a graph G is equitably k-colorable if there is a mapping $f : V(G) \to \{1, 2, \ldots, k\}$ such that $f(x) \neq f(y)$ whenever $xy \in E(G)$ and $||f^{-1}(i)| - |f^{-1}(j)|| \leq 1$ for $1 \leq i < j \leq k$. The equitable chromatic number of a graph G, denoted by $\chi_=(G)$, is the minimum k such that G is equitably k-colorable. The equitable chromatic threshold of a graph G, denoted by $\chi_=^*(G)$, is the minimum t such that G is equitably k-colorable for $k \geq t$. The current paper studies equitable chromatic numbers of Kronecker products of graphs. In particular, we give exact values or upper bounds on $\chi_=(G \times H)$ and $\chi_=^*(G \times H)$ when G and H are complete graphs, bipartite graphs, paths or cycles.

1. Introduction

All graphs considered in this paper are finite, undirected, loopless and without multiple edges. For a positive integer k, let $[k] = \{1, 2, \ldots, k\}$. A (*proper*) *k-coloring* of a graph G is a mapping $f : V(G) \to [k]$ such that $f(x) \neq f(y)$ whenever $xy \in E(G)$. We call the set $f^{-1}(i) = \{x \in V(G) : f(x) = i\}$ a *color class* for each $i \in [k]$. Notice that each color class is an independent set, i.e., a pairwise non-adjacent vertex set. A graph is *k-colorable* if it has a k-coloring. The *chromatic number* of G, denoted by $\chi(G)$, is equal to $\min\{k : G \text{ is } k\text{-colorable}\}$. An *equitable k-coloring* of G is a k-coloring for which any two color classes differ in size by at most one, or equivalently, each color class is of size $\lfloor |V(G)|/k \rfloor$ or $\lceil |V(G)|/k \rceil$. If G has n vertices, then the color classes of an equitable k-coloring have sizes $\lfloor (n + t - 1)/k \rfloor$ for $t \in [k]$. If we write $n = kq + r$ with $0 \leq r < k$, then exactly r (respectively, $k - r$) color classes have size $q + 1$ (respectively, q). The *equitable chromatic number* of G, denoted by $\chi_=(G)$, is equal to $\min\{k : G \text{ is equitably} k\text{-colorable}\}$, and the *equitable chromatic threshold* of G, denoted by $\chi_=^*(G)$, is equal to $\min\{t : G \text{ is equitably } k\text{-colorable for } k \geq t\}$. The *Kronecker* (or *cross, direct, tensor, weak tensor* or *categorical*) *product* of graphs G and H is the graph $G \times H$ with vertex set $V(G) \times V(H)$ and edge set $\{(x, y)(x', y') : xx' \in E(G) \text{ and } yy' \in E(H)\}$.

The concept of equitable colorability was first introduced by Meyer [25]. His motivation came from the problem of assigning one of the six days of the work week to each garbage collection route. For other applications such as scheduling and constructing timetables, please see [1,12,13,16,27,30,31]. We refer the reader to a survey given by Lih [23] for pertinent concepts and results.

In 1964, Erdős [7] conjectured that any graph G with maximum degree $\Delta(G) \leq k$ has an equitable $(k + 1)$-coloring, or equivalently, $\chi_=^*(G) \leq \Delta(G) + 1$. This conjecture was proved in 1970 by Hajnal and Szemerédi [9] with a long and complicated proof. Mydlarz and Szemerédi [26] found a polynomial algorithm for such a coloring. Recently, Kierstead and Kostochka [14] gave a short proof of the theorem, and presented another polynomial algorithm for such a coloring.

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They [15] proved an even stronger result that every graph satisfying $d(x) + d(y) \le 2k + 1$ for every edge xy has an equitable $(k + 1)$-coloring. Brooks' type results are conjectured: Equitable Coloring Conjecture [25] $\chi_=(G) \le \Delta(G)$, and Equitable Δ-Coloring Conjecture [4] $\chi^*_=(G) \le \Delta(G)$ for $G \notin \{K_n, C_{2n+1}, K_{2n+1,2n+1}\}$. Exact values of equitable chromatic numbers and equitable thresholds of trees [3] and complete multipartite graphs [2,22] were determined. Chen et al. [5] and Furmańczyk [8] investigated equitable colorability of square and cross products of graphs. Equitable coloring has been extensively studied in the literature, see [3,4,17–21,23,24,27,28,33–36].

Among the known results, we are most interested in those on graph products. Notice that graph products are engrossing that the purpose is not to construct a complex graph, but to decompose it into simple graphs. To study the relation of parameters between the product and its factors is helpful to analyze the structure of complicated graphs, see [10,11,29,32,37].

This paper is organized as follows. Section 2 is a review for equitable colorings on Kronecker products of graphs related to our results in this paper. Sections 3–6 establish exact values and upper bounds on equitable chromatic numbers and thresholds of Kronecker products of complete graphs, bipartite graphs, a long path or cycle with a complete graph, and P_2, P_3, C_3 or C_4 with a complete graph, respectively.

2. Preliminaries

For integer $n \ge 1$, the *n-path* P_n is the graph with vertex set $\{x_1, x_2, \ldots, x_n\}$ and edge set $\{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n\}$. For integer $n \ge 3$, the *n-cycle* C_n is the graph with vertex set $\{x_1, x_2, \ldots, x_n\}$ and edge set $\{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1\}$. For positive integers m and n, the *complete bipartite graph* $K_{m,n}$ is the graph with vertex set $\{y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_n\}$ and edge set $\{y_iz_j : i \in [m] \text{ and } j \in [n]\}$.

From the definitions, it is evident that $\chi(G) \le \chi_=(G) \le \chi^*_=(G)$ for any graph G. In general, the inequalities can be strict. As examples,

$$\begin{array}{llllll} \chi(K_{1,n}) & = 2 < \chi_=(K_{1,n}) & = & \chi^*_=(K_{1,n}) & = \left\lceil \frac{n+2}{2} \right\rceil & \text{for } n \ge 3, \\ \chi(K_{n,n}) & = & \chi_=(K_{n,n}) & = 2 < \chi^*_=(K_{n,n}) & = n+1 & \text{for odd } n \ge 3, \\ \chi(K_{3,3,6}) & = 3 < \chi_=(K_{3,3,6}) & = 4 < & \chi^*_=(K_{3,3,6}) & = 7. & \end{array}$$

Hedetniemi [10] has a famous conjecture for chromatic numbers, which is still open.

Conjecture 1 (*[10]*). $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$ *for any two graphs G and H.*

While it is easy to check that $\chi(G \times H) \le \min\{\chi(G), \chi(H)\}$, the inequality $\chi_=(G \times H) \le \min\{\chi_=(G), \chi_=(H)\}$ is false in general. For instance, Furmańczyk [8] gave that $\chi_=(P_3 \times P_3) = 3 > 2 = \min\{\chi_=(P_3), \chi_=(P_3)\}$. On the other hand, Chen et al. [5] gave the following result.

Lemma 2 (*[5]*). $\chi_=(G \times H) \le \min\{|V(G)|, |V(H)|\}$ *for any two graphs G and H.*

By Lemma 2 and Duffus–Sands–Woodrow's result [6] that $\chi(K_m \times K_n) = \min\{m, n\}$, Chen et al. [5] got that $\chi_=(K_m \times K_n) = \min\{m, n\}$. They also showed that $\chi_=(C_m \times C_n) = \chi^*_=(C_m \times C_n) = 2$ if mn is even, and 3 otherwise; and $\chi_=(K_n \times K_{n,n-1}) = \chi^*_=(K_n \times K_{n,n-1}) = n$. Furmańczyk [8] established that $\chi_=(K_m \times P_n) = 2$ if m is even or $n = 2$, and 3 otherwise; and $\chi_=(K_{1,m} \times K_{1,n}) = \min\{m, n\} + 1$.

In general, $\min\{|V(G)|, |V(H)|\}$ is not an upper bound of $\chi^*_=(G \times H)$. In fact, Chen et al. [5] gave a counterexample that $K_2 \times K_n$ is not equitably $(n + 1)/2$-colorable if $n > 1$ and $n \equiv 1 \pmod 4$. Also, $\chi^*_=(G \times H) \le \max\{\chi^*_=(G), \chi^*_=(H)\}$ is false in general. Two counterexamples given in [5] and [8], respectively, are $\chi^*_=(K_{2,3} \times K_{2,3}) = 3 > 2 = \chi^*_=(K_{2,3})$ and $\chi^*_=(P_3 \times P_3) = 3 > 2 = \max\{\chi^*_=(P_3), \chi^*_=(P_3)\}$. On the other hand, Chen et al. [5] gave the following conjecture.

Conjecture 3 (*[5]*). $\chi^*_=(G \times H) \le \max\{|V(G)|, |V(H)|\}$ *for any two graphs G and H.*

Notice that we only have to verify the conjecture for $K_m \times K_n$, and we show that it is true in Section 3.

3. Product of complete graphs

We first study equitable chromatic thresholds of Kronecker products of complete graphs. Our result gives a positive answer to Conjecture 3. We in fact give a slightly better upper bound.

Theorem 4. *For positive integers $m \le n$, we have* $\chi^*_=(K_m \times K_n) \le \left\lceil \frac{mn}{m+1} \right\rceil$.

Proof. We shall prove that $K_m \times K_n$ is equitably k-colorable for $k \ge \left\lceil \frac{mn}{m+1} \right\rceil$ by induction on $m + n$. The assertion is clear for $m = n = 1$, as $K_1 \times K_1 = K_1$. Suppose the assertion is true for $m' + n' < m + n$. Assume $mn = kq + r$, where $0 \le r < k$. Let $\sigma_t = \left\lfloor \frac{mn+t-1}{k} \right\rfloor$ for $1 \le t \le k$. Then $\sigma_i = q$ for $1 \le i \le k - r$ and $\sigma_j = q + 1$ for $k - r + 1 \le j \le k$. We consider three cases.

Case 1. $k \ge m + n$ and $r \ge m$. In this case, $\sigma_j = q + 1$ for at least m indices j's and $q + 1 = \frac{mn-r}{k} + 1 \le \frac{m(n-1)}{k} + 1 \le \frac{m}{m+n}(n - 1) + 1 < n$. Let $n' = n - q - 1$ and $k' = k - m$. Then $n' > 0$ and $k' \ge n \ge \max\{m, n'\} \ge \left\lceil \frac{mn'}{\min\{m,n'\}+1} \right\rceil$.