Discrete Applied Mathematics 158 (2010) 1838–1847

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Integer matrices with constraints on leading partial row and column sums

Stephen E. Wright *

Department of Statistics, Miami University, Oxford, OH 45056, United States

ARTICLE INFO

Article history:
Received 9 February 2009
Received in revised form 13 October 2009
Accepted 28 June 2010
Available online 22 July 2010

Keywords:
Integer linear programming
Consistent matrix rounding
Network flow
Totally unimodular matrix

ABSTRACT

Consider the problem of finding an integer matrix that satisfies given constraints on its leading partial row and column sums. For the case in which the specified constraints are merely bounds on each such sum, an integer linear programming formulation is shown to have a totally unimodular constraint matrix. This proves the polynomial-time solvability of this case. In another version of the problem, one seeks a zero–one matrix with prescribed row and column sums, subject to certain near-equality constraints, namely, that all leading partial row (respectively, column) sums up through a given column (respectively, row) are within unity of each other. This case admits a polynomial reduction to the preceding case, and an equivalent reformulation as a maximum-flow problem. The results are developed in a context that relates these two problems to consistent matrix rounding.

1. Introduction

Given positive integer vectors $r \in \mathbb{Z}^m$ and $c \in \mathbb{Z}^n$, the well-known theorem of Gale and Ryser [13,18] provides necessary and sufficient conditions for the existence of an $m \times n$ zero–one matrix X with row sums equaling the entries in r and column sums equaling the entries in c. In this paper we consider two related problems in which additional constraints are imposed on the leading partial row and column sums of X.

The first of these problems is to find a matrix $X \in \mathbb{Z}^{m\times n}$ satisfying the conditions

$$x_{ij}^- \le x_{ij} \le x_{ij}^+, \quad \text{for } i \in \{1, \dots, m\},\ j \in \{1, \dots, n\}, \tag{1}$$

$$c_{ij}^- \le \sum_{i'=1}^{i} x_{i'j} \le c_{ij}^+, \quad \text{for } j \in \{1, \dots, n\},\ i \in \{1, \dots, m\}, \tag{2}$$

$$r_{ij}^- \le \sum_{j'=1}^{j} x_{ij'} \le r_{ij}^+, \quad \text{for } i \in \{1, \dots, m\},\ j \in \{1, \dots, n\}, \tag{3}$$

where X^-, X^+, C^-, C^+, R^-, and R^+ are given $m \times n$ integer matrices. We shall refer to this as the *simple-bound problem*. The problem treated by the Gale–Ryser theorem is covered by the sub-case in which $X^- = 0$ and the bounds (2), (3) amount to equality constraints on row and column sums of the full matrix. This sub-case also includes various problems arising in discrete tomography [9,14], such as the problem of finding a zero–one matrix with prescribed line sums and a prescribed zero block [6,8]. Further motivation for the study of (1)–(3) arises through consideration of the problem described next.

* Fax: +1 5135291493.
E-mail address: wrightse@muohio.edu.

0166-218X/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2010.06.010

The second problem that we consider will be referred to as the *near-equality problem*. It consists of finding a zero–one matrix that, in addition to having prescribed row and column sums, also satisfies "near-equality" constraints on its leading partial row and column sums. Specifically, given $r \in \mathbb{Z}^m$ and $c \in \mathbb{Z}^n$, we wish to find a matrix $X \in \mathbb{Z}^{m\times n}$ satisfying the conditions

$$0 \le x_{ij} \le 1, \quad \text{for } i \in \{1,\dots,m\},\ j \in \{1,\dots,n\}, \tag{4}$$

$$\sum_{j=1}^{n} x_{ij} = r_i, \quad \text{for } i \in \{1,\dots,m\}, \tag{5}$$

$$\sum_{i=1}^{m} x_{ij} = c_j, \quad \text{for } j \in \{1,\dots,n\}, \tag{6}$$

$$\left|\sum_{j'=1}^{j} \left(x_{ij'} - x_{i'j'}\right)\right| \le 1, \quad \text{for } i, i' \in \{1,\dots,m\},\ j \in \{1,\dots,n\}, \tag{7}$$

$$\left|\sum_{i'=1}^{i} \left(x_{i'j} - x_{i'j'}\right)\right| \le 1, \quad \text{for } j, j' \in \{1,\dots,n\},\ i \in \{1,\dots,m\}, \tag{8}$$

or show that no such matrix exists. The constraints (4)–(6) are again of Gale–Ryser type and are therefore well understood. Note that (7) constrains the row sums for the leftmost $m \times j$ submatrix, while (8) constrains the topmost $i \times n$ submatrix. As we show in Section 3, however, these additional constraints do complicate matters: the full system (4)–(8) may be infeasible even though the simpler system (4)–(6) is feasible.

The near-equality conditions (7), (8) typically arise from a need for balance or uniformity in the distribution of ones throughout the matrix, so that blocks of similar size have a similar number of unit entries. For example, such a balancing requirement arises in connection with embedding grids in hypercubes [4,7], which in turn is motivated by the design of parallel computing architectures. The specific formulation given by (4)–(8) has been proposed in a particular approach [17] to such embeddings. The problem is also related to scheduling [5,20], in which case the rows and columns of the matrix correspond to tasks and stations (or times) at which they can be performed. The near-equality constraints reflect the goal of maintaining a balance across one or both dimensions. This can improve robustness against unforeseen machine failures or changes in demand for tasks, or it can provide a just-in-time schedule for sequential tasks. In both the embedding and scheduling applications, the precise order of the elements of the column-sum vector c may not be specified in advance, so the problem considered here might viewed as a subproblem to be solved repeatedly for various re-orderings of c.

The purpose of this paper is to demonstrate that the simple-bound and near-equality problems can be solved in polynomial time using methods from linear programming and/or network optimization. The key steps in this demonstration consist of building the following links: (a) the near-equality problem can be transformed into a special case of the simple-bound problem, (b) the simple-bound problem is closely related to consistent matrix rounding, and (c) constrained rounding in integer linear programming is related to total unimodularity and integrality of vertices in the simple-bound problem.

In the next section, we reformulate the simple-bound problem (1)–(3) as an integer linear program and invoke a result on matrix rounding to demonstrate that the constraint matrix is totally unimodular. This proves the polynomial-time solvability of the problem using, for example, linear programming methods. In Section 3, we derive valid inequalities for the near-equality problem (4)–(8). These yield a complete linear characterization, in the form of (1)–(3), of the polytope of integer solutions to (4)–(8). In Section 4, we use these valid inequalities to recast the near-equality problem in terms of finding a maximal flow in a network that is closely related to one introduced by Knuth [16] for matrix rounding. The final section (Section 5) shows that Knuth's network approach to rounding also provides an alternative, more elementary proof of the main integrality theorem of Section 2. In this way, the theory of matrix rounding and our two problems comes full circle and leads to multiple perspectives on all three.

2. Matrix rounding and integrality of the simple-bound problem

A key result in this paper is that every vertex of the polytope defined by the simple-bound problem (1)–(3) is integral. This yields a polynomial-time method for deciding whether (1)–(3) has an integer solution, because a vertex of such a polytope (or a certificate that the polytope is empty) can be found in polynomial time [19]. We provide two different proofs of the claimed integrality: one is given below and the other is delayed to Section 5.

In this section, we use the following result on consistent matrix rounding, which strengthens the classical result of Bacharach [3]. It was first proved by Knuth [16], although he stated the conclusion in a weaker form concerning only the full column and row sums, rather than the leading partial sums. Doerr et al. [11] later stated the result in its full generality.

Theorem 1. *For all $\hat{X} \in [0,1]^{m\times n}$, a rounding $X \in \{0,1\}^{m\times n}$ of $\hat{X}$ exists such that*