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Note

Complexity of 3-edge-coloring in the class of cubic graphs with a polyhedral embedding in an orientable surface☆

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ABSTRACT

A polyhedral embedding in a surface is one in which any two faces have boundaries that are either disjoint or simply connected. In a cubic (3-regular) graph this is equivalent to the dual being a simple graph. In 1968, Grünbaum conjectured that every cubic graph with a polyhedral embedding in an orientable surface is 3-edge-colorable. For the sphere, this is equivalent to the Four-Color Theorem, but we have disproved the conjecture in the general form. In this paper we extend this result and show that if we restrict our attention to a class of cubic graphs with a polyhedral embedding in an orientable surface, then the computational complexity of the 3-edge-coloring problem and its approximation does not improve.

1. Introduction

A *k-edge-coloring* of a graph is an assignment of numbers $1, \ldots, k$ to its edges such that any two incident edges receive different colors. By Tait [17], a cubic (3-regular) planar graph is 3-edge-colorable if and only if its geometric dual is 4-colorable. Thus the dual form of the Four-Color Theorem (see [1]) is that every 2-edge-connected planar cubic graph has a 3-edge-coloring.

Denote by $\mathcal{C}$ the class of cubic graphs. From the classical result of Vizing [19], it follows that there is a polynomial time algorithm which finds a 4-edge-coloring for any graph of $\mathcal{C}$ (see also [14]). On the other hand, by Holyer [8], the problem of deciding whether a graph from $\mathcal{C}$ is 3-edge-colorable is NP-complete.

For $G \in \mathcal{C}$, define by $\sigma(G)$ the minimum number such that there exists a 4-edge-coloring of G with $\sigma(G)$ edges assigned the fourth color. Define by $\rho(G)$ the minimum number of vertices that must be deleted from G to have the resulting graph be 3-edge-colorable. Clearly, $\sigma(G) = \rho(G) = 0$ if and only if G is 3-edge-colorable. In [15] it was proved that the problems of deciding whether $\rho(G), \sigma(G) \in [0, n^{1-\epsilon}]$ are NP-complete for $G \in \mathcal{C}$. Thus, it is an NP-hard problem to approximate $\rho(G)$ and $\sigma(G)$, $G \in \mathcal{C}$, with an error $O(n^{1-\epsilon})$.

An embedding of a cubic graph in a surface is called *polyhedral* if it is cellular (each face is homeomorphic to an open disk) and its dual is a simple graph (i.e., any two faces have at most one edge in common and the boundary of each face is a simple circuit; see, e.g., [2]). Let $\mathcal{P}$ denote the class of cubic graphs with a polyhedral embedding in an orientable surface (see, e.g., [3,6] for a formal definition of orientable surfaces). During a conference in 1968, Grünbaum [7] presented a conjecture that every graph from $\mathcal{P}$ is 3-edge-colorable. By the results of Appel and Haken [1] and Tait [17], the conjecture holds true for the sphere. A positive solution of this conjecture would generalize the dual form of the Four-Color Theorem to every orientable surface. We disproved the general form of the conjecture in [9,10].

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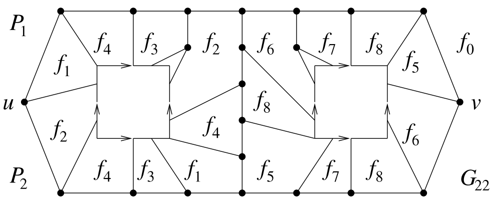

Fig. 1.

In this paper we continue with the study begun in [9,10] and show that 3-edge-coloring problems have the same computational complexity on classes $\mathcal{P}$ and $\mathcal{C}$. We generalize the results from [8,15] and prove that the problem of deciding whether a graph from $\mathcal{P}$ is 3-edge-colorable is NP-complete and the problems of deciding whether $\rho(G), \sigma(G) \in [0, n^{1-\epsilon}]$ are NP-complete for $G \in \mathcal{P}$.

2. Networks, flows and superposition

If G is a graph, then $V(G)$ and $E(G)$ denote the vertex and edge sets of G, respectively. If v is a vertex of G, then $\omega_G(v)$ denotes the set of edges having one end v and the other end from $V(G) \setminus \{v\}$.

By a *network* we mean a couple (G, U) where G is a graph and $U \subseteq V(G)$. By a *nowhere-zero* $\mathbb{Z}_2 \times \mathbb{Z}_2$*-flow* φ in (G, U) we mean a mapping $\varphi : E(G) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $\varphi(e) \neq 0$ for each edge e of G and $\partial\varphi(v) = \sum_{e \in \omega_G(v)} \varphi(e) = 0$ for each vertex $v \in V(G) \setminus U$. By a *nowhere-zero* $\mathbb{Z}_2 \times \mathbb{Z}_2$*-flow* in a graph G we mean a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow in $(G, \emptyset)$.

If G is a graph, then denote by $\rho_4(G)$ the minimum number n such that there exists $U \subseteq V(G)$, $|U| = n$, such that (G, U) has a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow (see [11]). In [12], we proved the following statement.

Lemma 1. *For every loopless cubic graph* G*,* $\rho_4(G) = \rho(G) = \sigma(G)$*.*

A graph is called *cyclically* k*-edge-connected* if deleting fewer than k edges does not result in a graph having at least two components containing cycles. Cubic graphs can have edge-connectivity at most 3, but arbitrarily large cyclic edge-connectivity. Denote by $\mathcal{C}_k$ and $\mathcal{P}_k$ the classes of graphs from $\mathcal{C}$ and $\mathcal{P}$ that are cyclically k-edge-connected, respectively.

Nontrivial cubic graphs without a 3-edge-coloring are called *snarks*. By nontrivial we mean cyclically 4-edge-connected and with girth (length of the shortest circuit) at least 5. The best known snark is the Petersen graph. Snarks present an important class of graphs, because the smallest counterexamples to many conjectures about graphs must be among them (see, e.g., [11,13]).

By a 4-*snark* we mean a network or a graph without a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow. It is well known that nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flows in a cubic graph G correspond to 3-edge-colorings of G by nonzero elements of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ (see, e.g., [11,13]). Thus snarks form a proper subclass of 4-snarks. In [11,13] we introduced a general method for constructing 4-snarks. It is based on the following two steps.

Suppose v is a vertex of a graph G and G' arises from G by the following process. Replace v by a graph H_v so that each edge e of G having one end v now has one end from H_v. If e is a loop and has both ends equal to v, then both ends of e will now be from H_v. Thus constructed, G' is called a *vertex superposition* of G.

Suppose e is an edge of G with ends u and v, and G' arises from G by the following process. Replace e by a graph H_e having at least two vertices, i.e., we delete e; pick two distinct vertices u', v' of H_e; and identify u' with u and v' with v. Then G' is called an *edge superposition* of G. Furthermore, if H_e is a 4-snark, then G' is called a 4-*strong* edge superposition of G.

We say that a graph G' is a (4-*strong*) *superposition* of G if G' arises from G after finitely many vertex and (4-strong) edge superpositions. In [11, Lemma 4.4], we proved that a 4-strong superposition of a 4-snark is a 4-snark. We use a stronger form of this statement, proved in [11, Proposition 8.1, items (b) and (c)].

Lemma 2. *If* G' *is a* 4*-strong superposition of* G*, then* $\rho_4(G') \geq \rho_4(G)$*.*

3. Reduction

In Fig. 1 there is depicted a snark G_{22} constructed in [9,10]. It is drawn in a plane with two handles. (A handle in a plane or a surface arises after deleting an open rectangle and identifying the opposite segments so that the orientations of the arrows, as indicated in Fig. 1, are preserved.) The boundary of the infinite face f_0 is a circuit C, which is composed from two paths P_1 and P_2 with ends u and v. By [10, items (4) and (5)], we have:

(1) any two faces f_i and f_j, $i, j \in \{1, \ldots, 8\}$, share at most one edge,
(2) the face f_0 shares exactly two edges with each f_i, $i \in \{1, \ldots, 8\}$, so P_1 and P_2 each contain exactly one of these edges.

In Fig. 2 is indicated a labeling of edges of G_{22} such that the edges incident with u or v have the same color and the edges incident with any other vertex have three different colors.

Lemma 3. *Suppose* G *is a connected cubic graph of order* n *and let* $d \geq 0$ *be an integer. Then we can construct in polynomial time a graph* $G_d \in \mathcal{P}_5$ *of order* $37^2 n + 2d$ *such that* $\rho_4(G_d) \geq \rho_4(G)$ *and* $\rho_4(G_d) = 0$ *if* $\rho_4(G) = 0$*.*