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Note

A polyhedral study of triplet formulation for single row facility layout problem

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ABSTRACT

The single row facility layout problem (SRFLP) is the problem of arranging n departments with given lengths on a straight line so as to minimize the total weighted distance between all department pairs. We present a polyhedral study of the triplet formulation of the SRFLP introduced by Amaral [A.R.S. Amaral, A new lower bound for the single row facility layout problem, Discrete Applied Mathematics 157 (1) (2009) 183–190]. For any number of departments n, we prove that the dimension of the triplet polytope is $n(n-1)(n-2)/3$ (this is also true for the projections of this polytope presented by Amaral). We then prove that several valid inequalities presented by Amaral for this polytope are facet-defining. These results provide theoretical support for the fact that the linear program solved over these valid inequalities gives the optimal solution for all instances studied by Amaral.

1. Introduction

In single row facility layout problem (SRFLP), the goal is to arrange n departments on a straight line. We are given the following data: an $n \times n$ symmetric matrix $C = [c_{ij}]$, where c_{ij} denotes the average daily traffic between two departments i and j, and the length l_i of each department $i \in N = \{1, \ldots, n\}$. The distance z_{ij} between two departments is considered to be the distance between their centroids. The objective is to find the permutation π that minimizes the total communication cost, i.e.

$$\min_{\pi} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{ij} z_{ij}^{\pi}.$$

The SRFLP has several applications involving arranging rooms on a corridor, machines in a manufacturing system, and books on a shelf [9,15,16]. The *minimum linear arrangement problem (MLAP)* was proven to be NP-hard in [8]. The SRFLP is a generalization of MLAP and so is also NP-hard. Numerous heuristic solution approaches have been proposed for SRFLP (e.g. see [9,12,17,14]).

Several exact solution techniques have also been proposed including branch and bound algorithms [16], dynamic programming [15,11], nonlinear programming [10], and linear mixed integer programming [1,2,13]. Anjos et al. [5] and Anjos and Vanelli [6] provided lower bounds on the optimal cost of SRFLP using semidefinite programming (SDP) relaxations. Anjos and Yen [7] computed near optimal solutions for instances with up to 100 facilities using a new SDP relaxation. Amaral and Letchford [4] conducted a polyhedral study on the distance polytope formulation of SRFLP and developed several classes of valid inequalities. They achieved quick bounds for SRFLP using LP relaxations based on these valid inequalities. They are comparable to the bounds achieved in [5].

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Amaral [3] presented an alternate formulation of the SRFLP, herein referred to as the *triplet formulation*, and introduced a set of valid inequalities for it. It is shown in [3] that the linear program solved over these valid inequalities yields the optimal solution for several classical SRFLP instances of sizes $n = 5$ to $n = 30$. These problem instances are from [1,2,9,10,13,16]. The results in [3] are comparable to the results of [6] which are based on SDP relaxation with cutting planes added.

The fact that the LP relaxation over the valid inequalities of [3] gives the optimal solution to so many instances suggests that these valid inequalities are quite strong. In this paper, we conduct a polyhedral study of the *triplet polytope*, i.e. the convex hull of feasible integer points for the triplet formulation. We prove that almost all valid inequalities introduced in [3] are indeed facet-defining for the triplet polytope. More specifically, we first show that the three polytopes (triplet polytope and its two projections defined in [3]) are of dimension $n(n-1)(n-2)/3$. After establishing the dimension of these polytopes, we then prove the aforementioned facet-defining properties.

The paper is organized as follows: Section 2 briefly reviews the triplet polytope, its projections, and the valid inequalities developed for them in [3]. In Section 3 we prove that these polytopes are of dimension $n(n-1)(n-2)/3$. In Section 4 we prove the facet-defining properties of valid inequalities of [3], and we conclude in Section 5 with a few remarks.

2. Triplet polytope, its projections and valid inequalities

In the triplet formulation for the SRFLP [3], a binary vector $\zeta \in \{0,1\}^{n(n-1)(n-2)}$ is used to represent a permutation of the departments in N. Each element of ζ is identified by a triplet subscript ijk, where $i, j, k \in N$ are distinct, and

$$\zeta_{ijk} = \begin{cases} 1 & \text{if department } k \text{ lies between departments } i \text{ and } j \\ 0 & \text{otherwise.} \end{cases}$$

Throughout the paper, all the department indices used in the subscript of a single variable, coefficient, or set are assumed to be distinct and we refrain from writing this in each case. We define

$$P = \left\{\zeta \in \{0,1\}^{n(n-1)(n-2)} : \zeta \text{ represents a permutation of } 1, \ldots, n\right\},$$

and refer to the convex hull of P, i.e. $conv(P)$, as the *triplet polytope*. Based on this formulation, the objective function of SRFLP can be written as

$$\min \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{ij} \left(\frac{1}{2}(l_i + l_j) + \sum_{k \neq i,\ k \neq j}^{n} l_k \zeta_{ijk} \right).$$

In [3] the following valid inequalities are presented for P:

$$0 \le \zeta_{ijk} \le 1 \quad i, j, k \in N \tag{1}$$

$$\zeta_{ijk} + \zeta_{ikj} + \zeta_{jki} = 1 \quad i, j, k \in N \tag{2}$$

$$\zeta_{ijd} + \zeta_{jkd} - \zeta_{ikd} \ge 0 \quad i, j, k, d \in N \tag{3}$$

$$\zeta_{ijd} + \zeta_{jkd} + \zeta_{ikd} \le 2 \quad i, j, k, d \in N. \tag{4}$$

Two projections of P are also introduced in [3]. We briefly review them here. It is clear that for any $\zeta \in P$

$$\zeta_{ijk} = \zeta_{jik} \quad 1 \le i < j \le n. \tag{5}$$

Using this identity, P can be projected onto the space $\{0,1\}^{n'}$, where $n' = n(n-1)(n-2)/2$. We refer to this projection as P^1. The projection of a vector $\zeta \in P$ will be a vector $\lambda \in P^1 \subseteq \{0,1\}^{n'}$ with elements λ_{ijk} such that $\lambda_{ijk} = \zeta_{ijk}$ for $i, j, k \in N$, $i < j$. So the valid inequalities (1)–(4) can also be projected yielding the following inequalities for P^1. Observe that (8)–(10) are obtained from projection of (3).

$$0 \le \lambda_{ijk} \le 1 \quad i, j, k \in N, i < j \tag{6}$$

$$\lambda_{ijk} + \lambda_{ikj} + \lambda_{jki} = 1 \quad i, j, k \in N,\ i < j < k \tag{7}$$

$$-\lambda_{ijd} + \lambda_{jkd} + \lambda_{ikd} \ge 0 \quad i, j, k, d \in N,\ i < j < k \tag{8}$$

$$\lambda_{ijd} + \lambda_{jkd} - \lambda_{ikd} \ge 0 \quad i, j, k, d \in N,\ i < j < k \tag{9}$$

$$\lambda_{ijd} - \lambda_{jkd} + \lambda_{ikd} \ge 0 \quad i, j, k, d \in N,\ i < j < k \tag{10}$$

$$\lambda_{ijd} + \lambda_{jkd} + \lambda_{ikd} \le 2 \quad i, j, k, d \in N,\ i < j < k. \tag{11}$$

Amaral [3] also introduces a more complicated set of valid inequalities for $conv(P^1)$ as follows: for a positive even integer $\beta \le n$, consider the set of distinct indices $S = \{i_t : t = 1, \ldots, \beta\} \subseteq \{1, \ldots, n\}$ and $d \in S$. Let (S_1, S_2) be a partition of $S \setminus \{d\}$ such that $|S_1| = \beta 2$. Then, the inequality

$$\sum_{p,q \in S_1 : p<q} \lambda_{pqd} + \sum_{p,q \in S_2 : p<q} \lambda_{pqd} \le \sum_{p \in S_h,\ q \in S_{\{1,2\} \setminus h} : h=1,2,\ p<q} \lambda_{pqd} \tag{12}$$

is valid for $conv(P^1)$ [3]. Inequalities (8)–(10) are special cases of (12) for $\beta = 4$, as noted in [3].