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Network flow interdiction on planar graphs

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ABSTRACT

The network flow interdiction problem asks to reduce the value of a maximum flow in a given network as much as possible by removing arcs and vertices of the network constrained to a fixed budget. Although the network flow interdiction problem is strongly NP-complete on general networks, pseudo-polynomial algorithms were found for planar networks with a single source and a single sink and without the possibility to remove vertices. In this work, we introduce pseudo-polynomial algorithms that overcome various restrictions of previous methods. In particular, we propose a planarity-preserving transformation that enables incorporation of vertex removals and vertex capacities in pseudo-polynomial interdiction algorithms for planar graphs. Additionally, a new approach is introduced that allows us to determine in pseudo-polynomial time the minimum interdiction budget needed to remove arcs and vertices of a given network such that the demands of the sink node cannot be completely satisfied anymore. The algorithm works on planar networks with multiple sources and sinks satisfying that the sum of the supplies at the sources equals the sum of the demands at the sinks. A simple extension of the proposed method allows us to broaden its applicability to solve network flow interdiction problems on planar networks with a single source and sink having no restrictions on the demand and supply. The proposed method can therefore solve a wider class of flow interdiction problems in pseudo-polynomial time than previous pseudo-polynomial algorithms and is the first pseudo-polynomial algorithm that can solve non-trivial planar flow interdiction problems with multiple sources and sinks. Furthermore, we show that the k-densest subgraph problem on planar graphs can be reduced to a network flow interdiction problem on a planar graph with multiple sources and sinks and polynomially bounded input numbers.

1. Introduction

In this paper, we are interested in minimizing the maximum flow of a network by removing arcs and vertices constrained to some interdiction budget. This problem is mainly known as *network interdiction* or *network flow interdiction*; sometimes the term *network inhibition* is used. One can either allow or disallow partial removal of arcs (removing half of an arc corresponds to reduce its capacity to half of the original value). However, the techniques and results do not substantially differ on this issue. We are interested in the case without partial arc removal. The problem of finding the k *most vital arcs* of a flow network is a special case of the network flow interdiction problem where k arcs have to be removed such that the maximum flow is reduced as much as possible. An important class of problems closely related to network interdiction is *stochastic network interdiction*. These are interdiction problems where one or more of the components of the network interdiction problem are not known with certainty. See [4,9] for more information on stochastic network interdiction.

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Network interdiction and related problems appear in various areas such as drug interdiction [20], military planning [8], protecting electric power grids against terrorist attacks [17] and hospital infection control [2]. The network flow interdiction problem was shown to be strongly NP-complete on general graphs and weakly NP-complete when restricted to planar graphs [15,20]. Different algorithms for finding exact solutions were proposed [8,13,16,20], which are mainly based on branch and bound procedures. In [3] a pseudo-approximation was presented. Earlier work includes [19].

When dealing with planar graphs with a single source and sink it was shown that by using planar duality, pseudo-polynomial algorithms for the network flow interdiction problem can be constructed when only arc removals are allowed [15]. Two of the major drawbacks of these algorithms (apart from the fact that they can only be applied on planar graphs) are the restrictions that vertex removals are not allowed and that the network must have exactly one source and one sink. Vertex removal can easily be formulated as arc removal by a standard technique of doubling vertices, and multiple sources and sinks are generally handled by the introduction of a supersource and supersink [1,6]. Unfortunately, these transformations destroy planarity and make it impossible to profit from the currently known specialized interdiction algorithms for planar graphs.

In this work, we are interested in the development of pseudo-polynomial algorithms for planar graphs that overcome various restrictions of previous methods. We propose a planarity-preserving transformation that enables incorporation of vertex removals and vertex capacities in pseudo-polynomial interdiction algorithms for planar graphs. We hereby answer a question raised in [15] asking how vertex capacities can be handled. The proposed algorithm can easily be transformed into a fully polynomial approximation scheme (FPAS) by using the rounding and scaling technique presented in [21].

Additionally, a pseudo-polynomial algorithm is introduced for the problem of determining the minimum interdiction budget needed to make it impossible to satisfy the demand of all sink nodes. The algorithm works on planar networks with multiple sources and sinks satisfying that the sum of the supplies at the sources equals the sum of the demands at the sinks. This problem is a generalization of the problem of determining whether a flow network is $n-k$ secure, i.e., any removal of k of its components does not impact the value of the maximum flow. A simple adaption of the proposed method allows us to broaden its applicability to solve interdiction problems on planar networks with a single source and sink without restriction on the demand and supply. The proposed method can therefore solve a wider class of interdiction problems in pseudo-polynomial time than previous pseudo-polynomial algorithms and is the first pseudo-polynomial algorithm that can solve non-trivial planar interdiction problems with multiple sources and sinks.

It is not known whether network flow interdiction on planar networks with multiple sources and sinks is a strongly NP-complete problem. To link the planar network flow interdiction problem with multiple sources and sinks to a more classical combinatorial problem we show that the k-densest subgraph problem on planar graphs can be reduced to a planar network flow interdiction problem with polynomially bounded numbers as input. However, it is not known if either of these problems can be solved in polynomial time.

The paper is organized as follows. We begin by giving some definitions and notations in Section 2. In Section 3, we give an overview of known complexity results on network flow interdiction and show how the k-densest subgraph problem on planar graphs can be reduced to a planar network flow interdiction problem with small input numbers. Section 4 presents an extension of currently known algorithms for network flow interdiction problems on undirected networks were only arc removals are allowed to the case of directed networks. We present in Section 5 a pseudo-polynomial algorithm for network flow interdiction on planar networks with a single source and sink that can handle vertex interdiction and vertex capacities. In Section 6 a pseudo-polynomial algorithm is presented that can be used for solving some network flow interdiction problems with multiple sources and sinks. Furthermore we show how the previously presented technique for modelling vertex interdiction and vertex capacities can be adapted to be used in the proposed algorithm for problems with multiple sources and sinks.

2. Preliminaries

2.1. Definitions and notations

Let (V, E) be a directed graph where V is the set of vertices, E is the set of arcs and for every arc $e \in E$, $u(e) \in \{0, 1, \ldots\}$ denotes its capacity. Two special nodes $s, t \in V, s \neq t$ designate the source node and sink node, respectively (the generalization to multiple sources and sinks is straightforward). We call the network $G = (V, E, u, s, t)$ a *flow network*. For $V', V'' \subseteq V$ we denote by (V', V'') the set of all arcs from V' to V''. Furthermore, for $V' \subseteq V$ we denote by $\omega^+(V')$ and $\omega^-(V')$ the set of all arcs exiting V' and entering V', respectively, i.e., $\omega^+(V') = (V', V \setminus V')$ and $\omega^-(V') = (V \setminus V', V')$. We also use the notation ω_G^+ and ω_G^- to specify the underlying graph G. For any subset V' of V, we denote by $G[V']$ the subgraph of G induced by V'. For any subset V' of V we denote by $[V', V \setminus V']$ the *cut* defined by V'. The *value of the cut* $[V', V \setminus V']$ is $\sum_{e\in\omega^+(V')} u(e)$. In the more general setting when every arc $e \in E$ has an additional lower bound $l(e)$ on the arc flow, the value of the cut $[V', V \setminus V']$ is defined by $\nu([V', V \setminus V']) = \sum_{e\in\omega^+(V')} u(e) - \sum_{e\in\omega^-(V')} l(e)$. The notation $\nu_G([V', V \setminus V'])$ is used to specify the underlying network G. A cut $[V', V \setminus V']$ in G is called *elementary* if $G[V']$ is connected. For two distinct vertices $s, t \in V$, a cut $[V', V \setminus V']$ is called an s–t cut if $s \in V', t \notin V'$.