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Three supplements to Reid's theorem in multipartite tournaments

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ABSTRACT

An n-partite tournament is an orientation of a complete n-partite graph. In this paper, we give three supplements to Reid's theorem [K.B. Reid, Two complementary circuits in two-connected tournaments, Ann. Discrete Math. 27 (1985) 321–334] in multipartite tournaments. The first one is concerned with the lengths of cycles and states as follows: let D be an $(\alpha(D)+1)$-strong n-partite tournament with $n \geq 6$, where $\alpha(D)$ is the independence number of D, then D contains two disjoint cycles of lengths 3 and $n-3$, respectively, unless D is isomorphic to the 7-tournament containing no transitive 4-subtournament (denoted by T_7^1). The second one is obtained by considering the number of partite sets that cycles pass through: every $(\alpha(D)+1)$-strong n-partite tournament D with $n \geq 6$ contains two disjoint cycles which contain vertices from exactly 3 and $n-3$ partite sets, respectively, unless it is isomorphic to T_7^1. The last one is about two disjoint cycles passing through all partite sets.

1. Terminology and introduction

We shall assume that the reader is familiar with the standard terminology on digraphs and refer the reader to [1]. All digraphs mentioned in this paper are finite without loops and multiple arcs. For a digraph D we denote by $V(D)$ and $A(D)$ the *vertex set* and *arc set* of D, respectively. If xy is an arc of a digraph D, then we say that x *dominates* y and write $x \to y$. More generally, for disjoint subsets A and B of $V(D)$ or subdigraphs of D, if every vertex of A dominates every vertex of B, then we say that A *dominates* B, denoted by $A \to B$. In the case when there is no arc from B to A, we write $A \Rightarrow B$. We also use the notation $A \leftrightarrow B$ to indicate that some vertex in A dominates some vertex in B and vice versa.

By a *cycle* and a *path* we mean a directed cycle and path, respectively. A k-*cycle* is a cycle of length k. If we consider a k-cycle $C = x_1x_2 \ldots x_kx_1$ in a digraph D, all subscripts appearing in related calculations are taken modulo the cycle length k (note that $x_0 = x_k$).

The *out-set* $N^+(x)$ of a vertex x is the set of vertices dominated by x, and the *in-set* $N^-(x)$ is the set of vertices dominating x. Every vertex of $N^+(x)$ is called an *out-neighbor* of x and every vertex of $N^-(x)$ is an *in-neighbor* of x. A digraph D is said to be *regular* if there is an integer k such that $|N^+(x)| = |N^-(x)| = k$ for every $x \in V(D)$.

If we replace every arc xy of a digraph D by yx, then we call the resulting digraph, denoted by D^{-1}, the *converse* digraph of D. For a subset A of $V(D)$, the subdigraph induced by A in D is denoted by $D\langle A\rangle$. Moreover, $D - A = D\langle V(D) - A\rangle$. For convenience, we define $D\langle A_1, A_2, \ldots, A_\ell\rangle$ to be the subdigraph induced by $A_1 \cup A_2 \cup \cdots \cup A_\ell \subseteq V(D)$.

A *strong component* H of D is a maximal subdigraph such that for any two vertices $x, y \in V(H)$, the subdigraph H contains a path from x to y and a path from y to x. The digraph D is *strong* or *strongly connected*, if it has only one strong component.

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D is k-*strong* if $|V(D)| \geq k+1$ and for any set X of at most $k-1$ vertices, the subdigraph $D - X$ is strong. The *connectivity* of D, denoted by $\kappa(D)$, is then defined to be the largest value of k such that D is k-strong.

An n-*tournament* is an orientation of a complete graph with n vertices. If a tournament contains no cycle, then it is called a *transitive* tournament. An n-*partite* or *multipartite tournament* is an orientation of a complete n-partite graph. If D is an n-partite tournament with the partite sets $V_1, V_2, \ldots, V_n$ such that $|V_1| \leq |V_2| \leq \cdots \leq |V_n|$, then $|V_n| = \alpha(D)$ is the *independence number* of D. In general, a digraph D is *cycle complementary* if there exist two disjoint cycles C_1 and C_2 such that $V(D) = V(C_1) \cup V(C_2)$ and the two cycles C_1 and C_2 are called *complementary cycles*.

There is an extensive literature about the existence of complementary cycles in digraphs. In 1985, Reid first derived a result to the problem of complementary cycles in tournaments.

Theorem 1.1 ([6]). *Every 2-strong tournament on $n \geq 6$ vertices contains two complementary cycles of lengths 3 and $n-3$, respectively, unless it is isomorphic to the 7-tournament containing no transitive 4-subtournament (we denote it by T_7^1).*

With Theorem 1.1 as the basic step, Song proved by induction the following.

Theorem 1.2 ([7]). *Let D be a 2-strong tournament on at least 6 vertices. Then, for every t satisfying $3 \leq t \leq n-3$, D contains two complementary cycles of lengths t and $n-t$, respectively, unless it is isomorphic to T_7^1.*

The problem of complementary cycles in multipartite partite tournaments is much more difficult to analyze than in tournaments. This is why up until now most of these results are about regular multipartite tournaments, see [5,9–11]. Recently, in the survey paper [12], Volkmann derived a nice result on complementary cycles in multipartite tournaments.

The aim of this article is to give some supplements to Theorem 1.1 in multipartite tournaments. The first one is concerned with the lengths of cycles and states as follows: every $(\alpha(D)+1)$-strong n-partite tournament D with $n \geq 6$ contains two disjoint cycles of lengths 3 and $n-3$, respectively, unless it is isomorphic to T_7^1. The second one is obtained by considering the number of partite sets that cycles pass through: every $(\alpha(D)+1)$-strong n-partite tournament D with $n \geq 6$ has two disjoint cycles which contain vertices from exactly 3 and $n-3$ partite sets, respectively, unless it is isomorphic to T_7^1. The last one confirms that every $(\alpha(D)+1)$-strong n-partite tournament D with $n \geq 6$ has two disjoint cycles C_1 and C_2 such that C_1 contains vertices from exactly 3 partite sets and $V(C_1) \cup V(C_2)$ contains at least one vertex from each partite set of D, unless it is isomorphic to T_7^1.

Moreover, some investigations in regular multipartite tournaments can also be derived from the main results of this article.

2. Preliminaries

The following results play an important role in our investigations.

Theorem 2.1 ([2]). *Each strong n-partite tournament contains an m-cycle for each $m \in \{3, 4, \ldots, n\}$.*

Theorem 2.2 ([8]). *If D is a non-strong n-partite tournament ($n \geq 2$) with the partite sets $V_1, V_2, \ldots, V_n$, then D can be decomposed into a unique sequence $D_1, D_2, \ldots, D_r$ such that $D_i \Rightarrow D_j$ for all $1 \leq i < j \leq r$ and there is at least one arc from D_t to D_{t+1} for $t = 1, 2, \ldots, r-1$, where D_ℓ is a strong component of D with $|V(D_\ell)| \geq 2$ or it is a subset of some partite set of D for $\ell = 1, 2, \ldots, r$.*

Remark 2.3. For an n-partite ($n \geq 2$) tournament D, whether it is strong or not, it can always be decomposed into a unique sequence $D_1, D_2, \ldots, D_r$ which satisfies the properties as described in Theorem 2.2, since if it is strong, then we can define the decomposition to be the only one component $D_1 = D$. We call $D_1, D_2, \ldots, D_r$ the *weak decomposition* of D and D_1 (D_r, respectively) the *initial component* (*terminal component*, respectively) of D.

Theorem 2.4 ([3]). *If D is a strong n-partite tournament, then every vertex of D is contained in a longest cycle of D.*

Corollary 2.5. *Every vertex of a strong n-partite tournament D is contained in a longest cycle of D which contains at least one vertex from each partite set.*

Theorem 2.6 ([4]). *Every vertex of a strongly connected n-partite tournament belongs to a cycle that contains vertices from exactly q partite sets for each $q \in \{3, 4, \ldots, n\}$.*

Theorem 2.7 ([13]). *If D is a multipartite tournament, then*

$$\kappa(D) \geq \left\lceil \frac{|V(D)| - 2i_\ell(D) - \alpha(D)}{3} \right\rceil,$$

where $i_\ell(D) = \max_{x \in V(D)} |d^+(x) - d^-(x)|$.