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On graphs determining links with maximal number of components via medial construction

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ABSTRACT

Let G be a connected plane graph, $D(G)$ be the corresponding link diagram via medial construction, and $\mu(D(G))$ be the number of components of the link diagram $D(G)$. In this paper, we first provide an elementary proof that $\mu(D(G)) \leq n(G) + 1$, where $n(G)$ is the nullity of G. Then we lay emphasis on the extremal graphs, i.e. the graphs with $\mu(D(G)) = n(G) + 1$. An algorithm is given firstly to judge whether a graph is extremal or not, then we prove that all extremal graphs can be obtained from K_1 by applying two graph operations repeatedly. We also present a dual characterization of extremal graphs and finally we provide a simple criterion on structures of bridgeless extremal graphs.

1. Introduction

In this paper, the graphs considered allow multiple edges and loops. For any graph G, let $p(G)$, $q(G)$ and $k(G)$ be the number of vertices, edges and connected components of G, respectively. The rank $r(G)$ and the nullity $n(G)$ of the graph G are defined to be $p(G) - k(G)$ and $q(G) - p(G) + k(G)$, respectively.

A graph is *planar* if it can be embedded in the plane, that is, it can be drawn on the plane so that no two edges intersect. A *plane* graph is a particular plane embedding of a planar graph. The different embeddings of a planar graph correspond to different plane graphs and they are all isomorphic to the abstract planar graph. Note that the nullity of a connected plane graph is equal to the number of bounded faces of the plane graph according to the well-known Euler formula. A *signed* graph is a graph with each edge labeled with a sign ($+$ or $-$); if it is also a plane graph, we call it a signed plane graph. A graph is said to be trivial if it is an isolated vertex without any edges.

A *knot* is a simple closed curve in Euclidean 3-space R^3, i.e. an embedding of S^1 into R^3. A *link* is the disjoint union of finite number of knots, each knot is called a *component* of the link. We denote by $\mu(L)$ the number of components of the link L. We take the convention that knot is a one-component link. In classical knot theory, one only considers tame links, that is, we can always think of closed curves as closed polygon curves. Although links live in Euclidean 3-space, we can always represent them by link diagrams, that is, regular projections with a short segment of the underpass curve cut at each double point of the projection.

There is a one-to-one correspondence between link diagrams and signed plane graphs via medial construction. We will give a brief exposition of the correspondence, and for the details and examples, see [11].

Given a non-trivial connected plane graph G, its medial graph $M(G)$ is defined as follows, see Chapter 17 *of* [3]. *The vertices of $M(G)$ are the edges of G. Each face $F = e_1, \ldots, e_r$ of length r in G determines r edges*

$$\{e_i e_{i+1} : 1 \leq i \leq r - 1\} \cup \{e_r e_1\}$$

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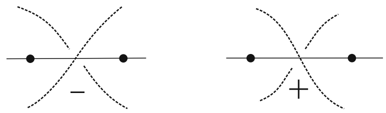

Fig. 1. The correspondence between a crossing and a signed edge.

of $M(G)$. In this definition, a loop e that bounds a face is viewed as a face of length one, and so determines one edge of $M(G)$, which is a loop on e. If G has an edge adjacent to a vertex of valency one, then the face containing that edge is viewed as having two consecutive occurrences of e and so once again there is a loop on e. If G is trivial, its medial graph is defined to be a simple closed curve surrounding the vertex (strictly, it is not a graph). If a plane graph G is not connected, its medial graph $M(G)$ is defined to be the disjoint union of the medial graphs of its connected components.

Given a signed plane graph G, we first draw its medial graph $M(G)$. To turn $M(G)$ into a link diagram $D(G)$, we turn the vertices of $M(G)$ into crossings by defining a crossing to be over or under according to the sign of the edge as shown in Fig. 1. Conversely, given a link diagram D, shade it as in a checkerboard so that the unbounded face is unshaded. Note that a link diagram can be viewed as a 4-regular plane graph and a 4-regular plane graph is 2-face-colorable, see Exercise 9.6.1 of [2]. Hence such a shading of D is always possible. We then associate D with a signed plane graph $G(D)$ as follows: For each shaded face F, take a vertex v_F, and for each crossing at which F_1 and F_2 meet, take an edge $v_{F_1}v_{F_2}$ and give the edge a sign also as shown in Fig. 1.

The following two facts will be obvious from the correspondence between signed plane graphs and link diagrams.

(1) The number of components of the link diagram (i.e. the number of components of the link it represents) corresponding to a signed plane graph is irrelevant to the signs of the edges of the graph. Hence, we will neglect the signs of the signed plane graph later.
(2) A connected plane graph and its dual graph correspond to the same medial graph, thus the numbers of components corresponding to a plane graph and its dual graph are the same.

In the figures that appear in the whole paper, we will use solid lines to represent the edges of plane graphs and dashed lines to represent the curves of their corresponding link diagrams.

The correspondence between link diagrams and signed plane graphs has been known for about one hundred years. Indeed, it provides a method of studying links using graphs. Originally it was used to construct a table of link diagrams of all links starting with graphs with a relatively small number of edges and then increasing the number of edges. In the late 1980s, the correspondence was used to obtain a relation between Jones polynomial [6] in knot theory and Tutte polynomial [14] in graph theory, see [7,8] for the details.

One of the first problems in studying links by using graphs via the correspondence may be determining the number of components of the link diagram corresponding to a plane graph via parameters of graphs. In this paper, we restrict ourself to connected plane graphs and study the number of components of their corresponding link diagrams. In Section 2, we will survey the known results in this aspect. In Section 3, we provide an upper bound for this number. Then we will lay emphasis on studying the extremal graphs, i.e. the graphs which reach the upper bound. An algorithm is given to judge whether a graph is extremal in Section 4. We prove that all extremal graphs can be obtained from K_1 by applying two graph operations repeatedly in Section 5. In Section 6, we present a dual characterization of extremal graphs, and in Section 7, we obtain a theorem, which characterizes the structure of bridgeless extremal graphs. We also obtain some simple necessary conditions for a bridgeless connected plane graph to be extremal in Section 7.

All proofs in the paper require only elementary knowledge of graph theory.

2. Some known results

The number of components of the link diagram corresponding to the plane graph G is also known as the number of straight-ahead walks of the medial graph of G [13], or the number of left–right cycles of the plane graph G, see Chapter 17 of [3].

The Tutte polynomial [14] $T_G(x, y)$ of a graph G contains a great deal of information about the graph, see Chapter 10 of [1] for a survey. It also plays an important role in determining the number of components of link diagrams. One has the following result [9]:

Theorem 2.1. *Let G be a connected plane graph, $T_G(x, y)$ be the Tutte polynomial of G and $\mu(D(G))$ be the number of components of the link diagram $D(G)$ corresponding to G. Then $T_G(-1, -1) = (-1)^{q(G)}(-2)^{\mu(D(G))-1}$.*

In [10], Mphako studied the number $T_G(-1, -1)$ and obtained the component numbers of link diagrams whose corresponding graphs are fans, wheels and wheels with q consecutive spokes missing. She also studied the component numbers of link diagrams corresponding to 2-sums of graphs.

Another result on $\mu(D(G))$ is related to the Laplacian matrix of the graph G. The Laplacian matrix $L(G)$ of a loopless graph G is defined as the matrix $L(G) = D(G) - A(G)$, where $D(G)$ is a diagonal matrix consisting of the degree of vertex v_i of the graph G in its iith entry, and $A(G)$ is the adjacency matrix of G. According to Theorem 17.3.5 and Lemma 14.15.3 of [3], we have: