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An upper bound for the restrained domination number of a graph with minimum degree at least two in terms of order and minimum degree

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ABSTRACT

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is a restrained dominating set if every vertex in $V - S$ is adjacent to a vertex in S and to a vertex in $V - S$. The restrained domination number of G, denoted $\gamma_r(G)$, is the smallest cardinality of a restrained dominating set of G. We will show that if G is a connected graph of order n and minimum degree δ and not isomorphic to one of nine exceptional graphs, then $\gamma_r(G) \leq \frac{n-\delta+1}{2}$.

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1. Introduction

For notation and graph theory terminology we in general follow [14]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order n and edge set E. For a set $S \subseteq V$, the subgraph induced by S in G is denoted by $\langle S \rangle$. If G_1 is an induced subgraph of G, then $G - G_1$ will denote the induced graph $\langle V(G) - V(G_1) \rangle$. The minimum degree (resp., maximum degree) among the vertices of G is denoted by $\delta(G)$ (resp., $\Delta(G)$). A one regular spanning subgraph of a graph G is called a *one factor* of G.

A set $S \subseteq V$ is a *dominating set* of G, denoted **DS**, if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted $\gamma(G)$, is the minimum cardinality of a **DS**. The concept of domination in graphs, with its many variations, is now well studied in graph theory. A thorough study of domination appears in [14,15].

A set **DS** $S \subseteq V$ is a *restrained dominating set*, denoted **RDS**, if every vertex in $V - S$ is adjacent to a vertex in $V - S$. Every graph has a restrained dominating set, since $S = V$ is such a set. The *restrained domination number* of G, denoted $\gamma_r(G)$, is the minimum cardinality of a **RDS** of G. Restrained domination was introduced by Telle and Proskurowski [17], albeit indirectly, as a vertex partitioning problem and further studied, for example, in [2–4,6,7,5,9–13,16,18].

One possible application of the concept of restrained domination is that of prisoners and guards. Here, each vertex not in the restrained dominating set corresponds to a position of a prisoner, and every vertex in the restrained dominating set corresponds to a position of a guard. Note that each prisoner's position is observed by a guard's position (to effect security) while each prisoner's position is seen by at least one other prisoner's position (to protect the rights of prisoners). To be cost effective, it is desirable to place as few guards as possible (in the sense above).

Let $\mathcal{B}$ be the set of graphs depicted in Fig. 1, and let $\mathcal{H}$ be the set of graphs G for which $\delta(G) \geq 2$ and at least one component is isomorphic to some $B_i \in \mathcal{B}$. Let $\mathcal{K}$ be the set of all even order complete graphs of order at least six with a one factor removed, and let $\mathcal{R} = \mathcal{H} \cup \mathcal{K}$.

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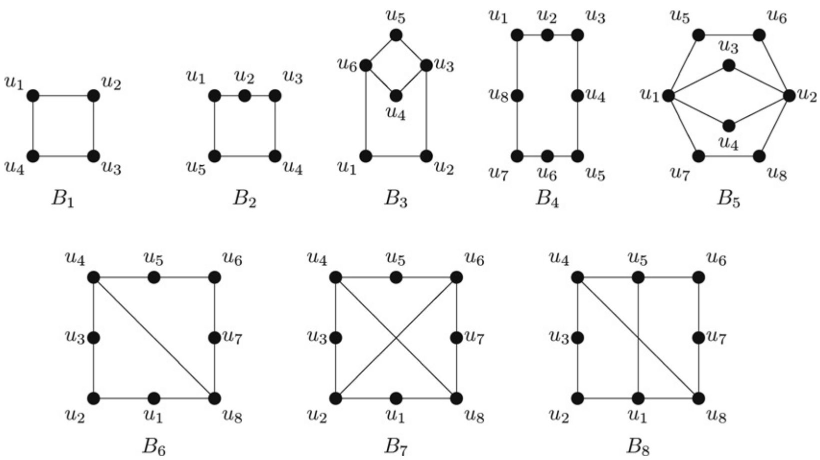

Fig. 1. The collection $\mathcal{B}$ of graphs.

The following result was proved in [7].

Theorem 1. *Let G be a connected graph with $\delta \geq 2$. If $G \notin \mathcal{B}$, then $\gamma_r(G) \leq \frac{n-1}{2}$.*

Berge [1] was the first to observe that $\gamma(G) \leq n - \Delta$, and graphs achieving this bound were characterized in [8]. Recently it was shown in [3] that if G is a connected graph of order n and $\delta \geq 2$, then $\gamma_r(G) \leq n - \Delta$. Thus, if $\delta \geq 2$, then $\gamma_r(G) \leq n - \delta$. Our aim in this paper is to improve the latter bound on the restrained domination number in terms of the order and minimum degree of the graph. We shall show:

Theorem 2. *Let G be a connected graph with $\delta \geq 2$. If $G \notin \mathcal{B} \cup \mathcal{K}$, then $\gamma_r(G) \leq \frac{n-\delta+1}{2}$.*

2. Proof of Theorem 2

We will employ induction on n. When $n = 3$, the graph G is a 3-cycle, and so $\gamma_r(G) = 1 = \frac{3-2+1}{2}$. This establishes the base case. For the inductive hypothesis, let $n \geq 4$ and assume that for all connected graphs G' with minimum degree $\delta' \geq 2$ of order n' with $3 \leq n' < n$ for which $G' \notin \mathcal{B} \cup \mathcal{K}$, we have $\gamma_r(G') \leq \frac{n'-\delta'+1}{2}$. Let G be a connected graph of order n such that $\delta \geq 2$ and $G \notin \mathcal{B} \cup \mathcal{K}$. If $\delta = 2$, the result is a consequence of Theorem 1. Thus, we also assume $\delta \geq 3$.

We begin with the following observations.

Observation 3. *Let H be a subgraph of G such that $\delta(H) \geq 2$ and $3 \leq n(H) < n(G)$. If $H \notin \mathcal{R}$, then $\gamma_r(H) \leq \frac{n(H)-\delta(H)+1}{2}$.*

Proof. Suppose $H \notin \mathcal{R}$. If H is connected, then $H \notin \mathcal{B} \cup \mathcal{K}$, and so, by the induction hypothesis, we have $\gamma_r(H) \leq \frac{n(H)-\delta(H)+1}{2}$.

Assume therefore that H is disconnected. Since $\delta(H) \geq 2$, each component of H has order at least three and size at least three. Suppose first that H has a component $C \in \mathcal{K}$. Then $\gamma_r(C) = 2$, while $n(C) = 2 + \delta(C) \geq 2 + \delta(H)$, and so the graph $C' = H - C$ has order at most $n(H) - \delta(H) - 2$. As $H \notin \mathcal{H}$, no component of C' is in $\mathcal{B}$, and so applying Theorem 1 to each component of C', we have $\gamma_r(C') \leq \frac{n(C')-1}{2}$. Hence $\gamma_r(H) \leq 2 + \frac{n(C')-1}{2} = 2 + \frac{n(H)-\delta(H)-3}{2} = \frac{n(H)-\delta(H)+1}{2}$.

We therefore assume that no component of H is in $\mathcal{K}$, and so no component of H is in $\mathcal{B} \cup \mathcal{K}$. If C is an arbitrary component of H, then $\delta(C) \geq \delta(H)$, and so, by the induction hypothesis, we have $\gamma_r(C) \leq \frac{n(C)-\delta(C)+1}{2} \leq \frac{n(C)-\delta(H)+1}{2}$. Moreover, by applying Theorem 1 to each component of $C' = H - C$, we have $\gamma_r(C') \leq \frac{n(C')-1}{2}$. Hence $\gamma_r(H) \leq \frac{n(C)-\delta(H)+1}{2} + \frac{n(C')-1}{2} = \frac{n(H)-\delta(H)}{2} < \frac{n(H)-\delta(H)+1}{2}$. $\diamond$

If the removal of some induced subgraph of G results in a disconnected graph, then $C(i)$ will denote the subgraph of G induced by all components which are isomorphic to B_i, $i = 1, \ldots, 8$.

Let G_1 be a nonempty induced subgraph of G and suppose $G_2 = G - G_1$ has $\delta(G_2) \geq 2$. Let S_1 be a **RDS** of G_1.

Observation 4. *If $V(C(i)) \neq \emptyset$ for some $i \in \{2, \ldots, 8\}$, then there exists a **RDS** of cardinality at most $|S_1| + \frac{n(G_2)-1}{2}$.*

Proof. Suppose that $V(C(i)) \neq \emptyset$ for some $i \in \{2, \ldots, 8\}$.

Suppose that a component of G_2 is isomorphic to C_5, and let $s \in V(C_5)$. If $V(G_2) - C(2) = \emptyset$, then G has a **RDS** of cardinality at most $|S_1| + \frac{2n(C(2))}{5} \leq |S_1| + \frac{n(G_2)-1}{2}$.

If $V(G_2 - C(2)) \neq \emptyset$, then the graph $G_2 - C(2)$ has, by Theorem 1, a **RDS** S_3 with cardinality at most $\frac{n(G_2)-n(C(2))}{2}$. If s is adjacent to a vertex in S_1, choose two vertices at distance two from s on C_5. If s is adjacent to a vertex in $V(G_1) - S_1$, choose

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