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Note

Approximation algorithms for art gallery problems in polygons ☆

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ABSTRACT

In this paper, we present approximation algorithms for minimum vertex and edge guard problems for polygons with or without holes with a total of n vertices. For simple polygons, approximation algorithms for both problems run in $O(n^4)$ time and yield solutions that can be at most $O(\log n)$ times the optimal solution. For polygons with holes, approximation algorithms for both problems give the same approximation ratio of $O(\log n)$, but the running time of the algorithms increases by a factor of n to $O(n^5)$.

1. Introduction

The art gallery problem is to determine the number of guards that are sufficient to *cover* or *see* every point in the interior of an art gallery. An art gallery can be viewed as a polygon P with or without holes with a total of n vertices and guards as points in P. Any point $z \in P$ is said to be visible from a guard g if the line segment joining z and g does not intersect the exterior of P. Usually the guards may be placed anywhere inside P. If the guards are restricted to vertices of P, we call them *vertex guards*. If there is no restriction, the guards are referred as *point guards*. Point and vertex guards are also referred as *stationary guards*. If the guards are mobile, i.e., able to patrol along a segment inside P, they are called *mobile guards*. If the mobile guards are restricted to edges of P, they are called *edge guards*.

The art gallery problem was first posed by Victor Klee for stationary guards (see [24]). Chvátal [9] proved that a simple polygon P needs at most $\lfloor n/3 \rfloor$ stationary guards. Fisk [18] later gave a simple proof of this result using coloring technique, and based on his proof, Avis and Toussaint [3] developed an $O(n \log n)$ time algorithm for positioning guards in P. O'Rourke [35] showed that P needs at most $\lfloor n/4 \rfloor$ mobile guards. For edge guards, $\lfloor n/4 \rfloor$ edge guards seem to be sufficient for guarding P, except for a few polygons (see [42]).

For a simple orthogonal polygon P, i.e., the edges of P are horizontal or vertical, Kahn et al. [26] proved that P needs at most $\lfloor n/4 \rfloor$ stationary guards. O'Rourke [34] later gave an alternative proof for this result. These proofs use the partition of P into convex quadrilaterals before $\lfloor n/4 \rfloor$ guards are placed in P. Note that a convex quadrilaterization of P can be obtained by algorithms of Edelsbrunner, O'Rourke and Welzl [12], Lubiw [32], Sack [38], and Sack and Toussaint [39]. Aggarwal [1] showed that P needs at most $\lfloor \frac{3n+4}{16} \rfloor$ mobile guards. This bound also holds for edge guards as shown by Bjorling-Sachs [6].

For a polygon P with h holes, O'Rourke [36] showed that P needs at most $\lfloor \frac{n+2h}{3} \rfloor$ vertex guards. Hoffmann, Kaufmann and Kriegel [23] and Bjorling-Sachs and Souvaine [7] proved independently that P can always be guarded with $\lceil \frac{n+h}{3} \rceil$ point

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guards. Bjorling-Sachs and Souvaine also gave an $O(n^2)$ time algorithm for positioning the guards. There is no tight bound known on the number of mobile guards required to guard P. Since $\lceil \frac{n+h}{3} \rceil$ point guards are sufficient to guard P, the bound naturally holds for mobile guards as well. To guard an orthogonal polygon P with h holes, Györi, Hoffmann, Kriegel and Shermer [22] proved that $\lfloor \frac{3n+4h+4}{16} \rfloor$ mobile guards are always sufficient to guard P. For survey of art gallery theorems and algorithms, see Ghosh [20], O'Rourke [36], Shermer [41] and Urrutia [42].

The minimum guard problem is to find the minimum number of guards that can see every point in the interior of a polygon. O'Rourke and Supowit [37] showed that the minimum vertex, point and edge guard problems in polygons with holes are NP-hard. Even in the case of polygons without holes, Lee and Lin [30] showed that the minimum vertex, point and edge guard problems are NP-hard. The minimum vertex and point guard problems are also NP-hard for simple orthogonal polygons as shown by Katz and Rpoisman [27] and Schuchardt and Hecker [40].

In this paper, we present approximation algorithms for minimum vertex and edge guard problems for polygons with or without holes. The approximation algorithms partition the polygonal region into convex components and construct sets consisting of these convex components. Then the algorithms use an approximation algorithm for the minimum set-covering problem on these constructed sets to compute the solution for the minimum vertex and edge guard problems. For simple polygons, approximation algorithms for both problems run in $O(n^4)$ time and yield solutions that can be at most $O(\log n)$ times the optimal solution. For polygons with holes, approximation algorithms for both problems give the same approximation ratio of $O(\log n)$, but the algorithms take $O(n^5)$ time.

It may be noted that approximation algorithms presented in preliminary versions of this paper run in $O(n^5 \log n)$ time for polygons with or without holes. The improvement in the running time of approximation algorithms is due to the improvement in the upper bound on the number of convex components in the convex partition of a polygon. There is no change in the method of transforming art gallery problems into set cover problems for computing vertex or edge guards in both types of polygons. For the last two decades, this is the only known technique for transforming these four art gallery problems leading to efficient approximation algorithms in terms of worst-case running times and approximation bounds.

Recently, Efrat and Har-Peled [13] presented randomized approximation algorithms for the minimum vertex guard problem in polygons. For simple polygons P, the randomized approximation algorithm runs in $O(nc_{\text{opt}}^2 \log^4 n)$ expected time and the approximation ratio is $O(\log c_{\text{opt}})$, where c_{opt} is the number of vertices in the optimal solution. In the worst case, c_{opt} can be a fraction of n. For polygons P with h holes, the randomized approximation algorithm runs in $O(nhc_{\text{opt}}^3 \textit{polylog}\ n)$ expected time and the approximation ratio is $O(\log n \log(c_{\text{opt}} \log n))$. Note that their randomized approximation algorithms do not always guarantee solutions and the quality of approximation is correct with high probability. No other approximation algorithm (deterministic or randomized) is known for the minimum vertex or edge guard problem in polygons. However, for special classes of polygons, there are approximation algorithms for the minimum point guard problem [33]. Also, there are approximation algorithms for the minimum vertex and point guard problems in 1.5-dimensional and 2.5-dimensional terrains [5,11,14,16,21,27,28].

In the next section, we present approximation algorithms for the minimum vertex guard problem. In Section 3, we present approximation algorithms for the minimum edge guard problem. In Section 4, we conclude the paper with a few remarks.

2. Approximation algorithms for vertex guards

Assume that vertices of the given polygon P are labeled $v_1, v_2, \ldots, v_n$. Let $VP(P, z)$ denote the set of all points of P that are visible from a point $z \in P$. If z is a vertex of P (say, v_i), then $VP(P, v_i)$ is called *fan* (say, F_i) and v_i is called the *fan vertex* of F_i. Otherwise, $VP(P, z)$ is called the *visibility polygon* of P from z. Since the region of P that can be seen by a vertex guard is a fan, the vertex guard problem of P can be view as a polygon decomposition problem in which pieces of the decomposition are fans.

It appears that if the entire boundary of P is visible from vertex guards, then the guards can also see every point in the interior of P. In Fig. 1(a), vertices v_7, v_{12} and v_{17} together can see the entire boundary of P, but the shaded region is not visible from any of them. This establishes that vertex guards must be chosen in such a way so that all boundary points as well as all internal points of P are visible from the chosen guards. In our approximation algorithms, the region of P is decomposed into a set of convex pieces and each piece lies at least in one of the chosen fans so that the entire region of P is covered.

It seems natural to restrict convex pieces in a polygon to be bounded by extensions of polygonal edges. Feng and Pavlidis [17] argued that this is a very natural restriction for polygonal decomposition problems in syntactic pattern recognition. In Fig. 1(b), three fans with fan vertices v_1, v_4 and v_7 are necessary to cover the polygon if only edge extensions are allowed, whereas two fans with fan vertices v_1 and v_7 suffice if boundaries of convex pieces are bounded by segments passing through any two vertices of the polygon. So, the polygonal region is decomposed into convex pieces where every component is bounded by segments that contains two vertices of the polygon.

A convex region $c \subset P$ is said to be a *convex component* of P if there is no other convex region b of P, where $c \subset b$, such that b can be divided by a line segment passing through two vertices of P. For the vertex guard problem, this restriction turns out to be a true restriction, as shown in the following lemma.

Lemma 2.1. *Every convex component of P is either totally visible or totally not visible from a vertex of P.*