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On probe permutation graphs

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ABSTRACT

Given a class of graphs $\mathcal{G}$, a graph G is a *probe graph of* $\mathcal{G}$ if its vertices can be partitioned into two sets, $\mathbb{P}$, the *probes*, and an independent set $\mathbb{N}$, the *nonprobes*, such that G can be embedded into a graph of $\mathcal{G}$ by adding edges between certain vertices of $\mathbb{N}$. If the partition of the vertices into probes and nonprobes is part of the input, then we call the graph a *partitioned probe graph of* $\mathcal{G}$. In this paper, we provide a recognition algorithm for partitioned probe permutation graphs with time complexity $O(n^2)$, where n is the number of vertices of the input graph. We show that a probe permutation graph has at most $O(n^4)$ minimal separators. As a consequence, for probe permutation graphs there exist polynomial-time algorithms solving problems like TREEWIDTH and MINIMUM FILL-IN. We also characterize those graphs for which the probe graphs must be weakly chordal.

1. Introduction

Chromosomes are very long DNA sequences. To find the sequence of a chromosome, biologists use enzymes to cut it into relatively small fragments, called *clones*. Different enzymes cut the chromosome in different ways; so within a large library of clones many clones will overlap. However the order of the clones is lost. In DNA *physical mapping*, one wishes to find the linear order of the clones based upon experimental information. To save some experimental cost to test the overlap between clones, the following algorithm is proposed by Zhang et al. [18,22]. The clones are distinguished as being either probes or nonprobes. No experiments are performed to test whether pairs of nonprobes overlap, but each probe is tested against each other probe and each nonprobe to determine whether they overlap. The DNA physical mapping can then be completed based upon the incomplete overlapping information.

In graph terminology, we are given a graph G whose vertices are distinguished as being either probes or nonprobes. The set of all nonprobes is an independent set of G. We want to construct a graph H having some given property π by adding certain edges to G between vertices identified as nonprobes. If it is possible, we call G a *partitioned probe* π *graph.* A graph G is called an *interval graph* if it has the following property: each vertex $v \in V(G)$ can be assigned a real interval I_v such that (x, y) is an edge of G if and only if $I_x \cap I_y \neq \emptyset$. Therefore, we can formulate the above DNA physical mapping based upon incomplete overlapping information as *the partitioned probe-interval-graph recognition problem.*

Let $\mathcal{G}$ be a class of graphs. A graph $G = (V, E)$ is a *probe graph of* $\mathcal{G}$ if its vertices can be partitioned into a set $\mathbb{P}$ of probes and an independent set $\mathbb{N}$ of nonprobes such that G can be embedded into a graph G' of $\mathcal{G}$ by adding certain edges between

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1 Resident at Institute of Mathematics, Academia Sinica, Taipei, Taiwan, when this research was done.

2 It is with deep sadness that we report the death of our friend Jiping (Jim) Liu on 14 January 2006, as the result of an automobile accident.

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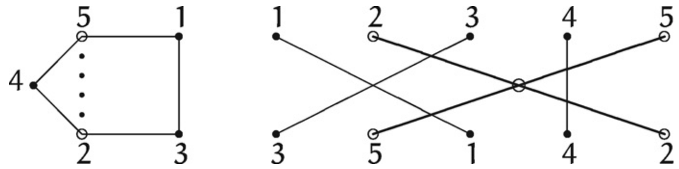

Fig. 1. A 5-cycle. Vertices 2 and 5 are nonprobes.

vertices of $\mathbb{N}$. We call G' an *embedding* of G. If the partition of the vertices of G into a set $\mathbb{P}$ and a set $\mathbb{N}$ is a part of the input, we refer to the graph as a *partitioned probe graph of* $\mathscr{G}$ and we denote such a graph as $G = (\mathbb{P} + \mathbb{N}, E)$. According to [1], probe chordal graphs also find immediate applications, *e.g.*, in the reconstruction of phylogenies. We think that the study of probe graph classes is of great interest since, first of all, it establishes demarcations on the *robustness* of a graph class with respect to irresolute inputs. The concept of probe graph classes was contemplated initially for this purpose. It also brings to light many interesting, sometimes unforeseen, properties of the new graph class in question. For example, it turns out that probe chordal graphs are perfect [11]. In fact, the same is true for *all* classes of *Meyniel graphs* [13]. That is, the probe graphs of a graph class remain perfect if and only if the class is a subclass of Meyniel graphs.

Another example is that probe interval graphs and probe distance-hereditary graphs are subclasses of *weakly chordal graphs.* In Section 5, we characterize all the graphs whose probe graphs must be weakly chordal. It is easy to see that the clique number remains solvable in polynomial time for partitioned probe perfect graphs and that the chromatic number is at most one more than the clique number in those graphs [6]. This fact by itself provides motivation to study probe graphs of classes of perfect graphs.

In this paper, we apply a modular decomposition technique for the recognition of partitioned probe permutation graphs. The recognition of the unpartitioned case remains an open problem. We conjecture that it is polynomially solvable.

Probe permutation graphs are in general not perfect (see Fig. 1), but they have many other interesting features. In Section 4, we prove that a probe permutation graph has at most $O(n^4)$ minimal separators, where n is the number of its vertices. An algorithm to find all minimal separators in a graph with polynomial delay appeared in [16]. As a consequence, there exist polynomial-time algorithms solving problems like TREEWIDTH and MINIMUM FILL-IN for probe permutation graphs [4]. Note that the treewidth and pathwidth parameters coincide for permutation graphs [16]. Thus the PATHWIDTH problem, which in general is much more difficult to compute, can also be solved in polynomial time for permutation graphs. It is easy to see that pathwidth and treewidth do not coincide for probe permutation graphs in general.

2. Preliminaries

A graph G is a pair (V, E), where the elements of V are called the *vertices* of G and where E is a family of two-element subsets of V, called the *edges*. We let n and m be the numbers of vertices and edges of a graph G, respectively. For a vertex x we write $N(x)$ for its set of neighbors in G, and $N[x] = N(x) \cup \{x\}$. For a subset $W \subseteq V$ we write $N(W) = \cup_{x \in W} N(x) \setminus W$, $N[W] = \cup_{x \in W} N[x]$, and $G[W]$ for the subgraph of G *induced* by W. For convenience, we write $G - W$ for the graph $G[V - W]$, i.e., the subgraph induced by $V - W$. For a vertex x we write $G - x$ rather than $G - \{x\}$. For two sets A and B we write $A + B$ and $A - B$ instead of $A \cup B$ and $A \setminus B$ respectively. For an element x we write $A - x$ instead of $A - \{x\}$ and $A + x$ instead of $A \cup \{x\}$.

Let π be a permutation of $(1, \ldots, n)$. The *matching diagram* of π is obtained as follows. Write the integers $(1, \ldots, n)$, horizontally from left to right. Underneath, write the integers $(\pi_1, \ldots, \pi_n)$, also horizontally from left to right. Draw n straight line segments connecting the two 1's, the two 2's, and so on. A graph is a *permutation graph* if it is isomorphic to the intersection graph of the line segments of a matching diagram.

Let $x \to y$ denote a directed edge from x to y. A graph G is a *comparability graph* if the edges of G can be directed such that the directed edges of resulting digraph satisfies the transitive property, i.e., $x \to y$ and $y \to z$ imply that $x \to z$. Let $\overline{G}$ denote the complement of G.

Theorem 1 *([21]). A graph is a permutation graph if and only if G and $\overline{G}$ are comparability graphs.*

Notice that permutation graphs form a *self-complementary* class of graphs. That is, if a graph is a permutation graph then so is its complement $\overline{G}$. The class of probe permutation graphs, however, is *not* self-complementary. For example, the disjoint union of two disjoint 6-cycles becomes a permutation graph if one long diagonal is added to each C_6 (Fig. 2). In the complement of $2C_6$, the nonprobes necessary to make each $\overline{C}_6$ subgraph a probe permutation graph are not independent in the combined graph. Therefore we introduce the following concept.

Definition 2 *([6]).* Let $G = (\mathbb{P} + \mathbb{N}, E)$ be a partitioned graph. The *sandwich conjugate* $G^* = (\mathbb{P} + \mathbb{N}, E')$ of G is the partitioned graph obtained from $\overline{G}$ by removing all edges between vertices of $\mathbb{N}$.

Note that, if $\mathscr{G}$ is a self-complementary class of graphs, then G is a partitioned probe graph of $\mathscr{G}$ if and only if its sandwich conjugate falls into the same category.

Theorem 1 permits permutation graphs to be recognized in linear time, using the linear-time algorithm of [17] to find a *modular decomposition tree.*