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Minimal split completions[☆]

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ABSTRACT

We study the problem of adding an inclusion minimal set of edges to a given arbitrary graph so that the resulting graph is a split graph, called a minimal split completion of the input graph. Minimal completions of arbitrary graphs into chordal and interval graphs have been studied previously, and new results have been added recently. We extend these previous results to split graphs by giving a linear-time algorithm for computing minimal split completions. We also give two characterizations of minimal split completions, which lead to a linear time algorithm for extracting a minimal split completion from any given split completion.

We prove new properties of split graph that are both useful for our algorithms and interesting on their own. First, we present a new way of partitioning the vertices of a split graph uniquely into three subsets. Second, we prove that split graphs have the following property: given two split graphs on the same vertex set where one is a subgraph of the other, there is a sequence of edges that can be removed from the larger to obtain the smaller such that after each edge removal the modified graph is split.

1. Introduction

Split graphs are the class of graphs whose vertices can be partitioned into an independent set and a clique; a graph class which is well studied and of wide theoretical interest and use [8]. Any graph can be embedded into a split graph by adding edges, and the resulting split graph is called a *split completion* of the input graph. A *minimum* split completion is a split completion with the minimum number of edges, and computing such split completions is an NP-hard problem [18]. A split completion H of a given graph G is *minimal* if no proper subgraph of H is a split completion of G. In this paper we show that a minimal split completion of a given graph can be computed in linear time.

Minimum and minimal chordal completions, also called *triangulations*, and minimum and minimal interval completions are defined analogously, by replacing split with chordal and with interval. Computing a minimum triangulation and computing a minimum interval completion of a graph are NP-hard problems [7,21], whereas it was shown already in 1976 that minimal triangulations can be computed in polynomial time [20]. In the mid-1990s minimal triangulations began to be studied again, and new characterizations have been given [5,15,19], which made new algorithms possible. Since then, the interest in minimal completion problems has increased, which has led to faster algorithms for minimal triangulations [16,17,13] and the knowledge that minimal interval completions can be computed and characterized in polynomial time [11,12]. Minimal split completions have not been studied earlier, and with this paper we expand the knowledge about classes of graphs into which minimal completions of arbitrary graphs can be computed in polynomial time.

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Minimal triangulations are well studied, and several characterizations of them have been given [10]. An algorithmically useful characterization is that a triangulation is minimal if and only if no single fill edge can be removed without destroying chordality of the triangulation [20] (fill edges are the edges added to the original graph to obtain a completion). This property does not hold for minimal interval completions. In this paper, we show that it holds for minimal split completions. In fact, we show the following more general result on split graphs: Between a split graph $G_1 = (V, E_1)$ and a split graph $G_2 = (V, E_2)$ with $E_1 \subset E_2$, there is a sequence of split graphs that can be obtained by repeatedly removing one single edge from the previous split graph, starting from G_2. This result is known to be true for chordal graphs [20,1], and it has been posed as an open problem for chordal bipartite graphs [2]. We characterize the fill edges that are candidates for removal when a non-minimal split completion H of an arbitrary graph G is given. Based on this, we give linear-time algorithms both for computing minimal split completions, and for removing edges from a given split completion to obtain a minimal split completion. In order to obtain our results, we define a new way of partitioning the vertices of a given split graph, called a *3-partition*. This partition is always unique, which is not the case for the traditional way of partitioning the vertices of a split graph into an independent set and a clique.

This paper is organized as follows. In the next section we give the necessary graph theoretical background, assuming that the reader is familiar with the basic notions of graph theory. In Section 3 we present our results on split graphs sandwiched between two given split graphs, and use these results to characterize minimal split completions. A new way of partitioning the vertices of a split graph uniquely is presented in Section 4, and this is used to give another characterization of minimal split completions in Section 5. These results are then combined to give an algorithm for removing redundant fill edges from a split completion to obtain a minimal split completion in Section 6, and an algorithm for directly computing a minimal split completion of an arbitrary input graph in Section 7. We conclude with some discussions and examples in Section 8.

2. Definitions and background

All graphs in this paper are simple and undirected. For a graph $G = (V, E)$, we let $n = |V|$ and $m = |E|$. The set of neighbors of a vertex $v \in V$ is denoted by $N(v)$, and the degree of a vertex v is denoted by $d(v) = |N(v)|$. We distinguish between subgraphs and induced subgraphs. In this paper, a *subgraph* of $G = (V, E)$ is a graph $G_1 = (V, E_1)$ with $E_1 \subseteq E$, and a *supergraph* of G is a graph $G_2 = (V, E_2)$ with $E \subseteq E_2$. We will denote these relations informally by the notation $G_1 \subseteq G \subseteq G_2$ (proper subgraph relation is denoted by $G_1 \subset G$). The complement of G is denoted by $\bar{G}$.

A simple cycle on k vertices is denoted by C_k and a complete graph on k vertices is denoted by K_k. Thus $2K_2$ is the graph that consists of 2 isolated edges. A graph is *chordal* if it contains no induced simple cycle of length at least 4. A subset K of V is a *clique* if K induces a complete subgraph of G. A subset I of V is an *independent set* if no two vertices of I are adjacent in G. We use $\omega(G)$ to denote the size of a largest clique in G, and $\alpha(G)$ to denote the size of a largest independent set in G.

G is a *split graph* if there is a partition $V = I + K$ of its vertex set into an independent set I and a clique K. Such a partition is called a *split partition* of G. There is no restriction on the edges between vertices of I and vertices of K. The partition of a split graph into a clique and an independent set is not necessarily unique. The following theorem from [9] states the possible partition configurations.

Theorem 1 (*Hammer and Simeone [9]*). *Let G be a split graph whose vertices have been partitioned into an independent set I and a clique K. Exactly one of the following conditions holds:*

(i) $|I| = \alpha(G)$ *and* $|K| = \omega(G)$
(the partition $I + K$ is unique).
(ii) $|I| = \alpha(G)$ *and* $|K| = \omega(G) - 1$
(there exists a vertex $x \in I$ such that $K \cup \{x\}$ is a clique).
(iii) $|I| = \alpha(G) - 1$ *and* $|K| = \omega(G)$
(there exists a vertex $y \in K$ such that $I \cup \{y\}$ is an independent set).

The following theorem characterizes split graphs, and we will use condition (iii) to prove one of the characterizations of minimal split completions that we present.

Theorem 2 (*Földes and Hammer [6]*). *Let G be an undirected graph. The following conditions are equivalent:*

(i) *G is a split graph.*
(ii) *G and $\bar{G}$ are chordal graphs.*
(iii) *G contains no induced subgraph isomorphic to $2K_2$, C_4 or C_5.*

Remark 3. Every induced subgraph of a split graph is also a split graph.
Graph classes satisfying this property are called hereditary.

For a given arbitrary graph $G = (V, E)$, a split graph $H = (V, E \cup F)$, with $E \cap F = \emptyset$, is called a *split completion* of G. The edges in F are called *fill edges*. H is a *minimal* split completion of G if $(V, E \cup F')$ fails to be a split graph for every proper subset F' of F.