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The stable marriage problem with master preference lists

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Abstract

We study variants of the classical stable marriage problem in which the preferences of the men or the women, or both, are derived from a *master* preference list. This models real-world matching problems in which participants are ranked according to some objective criteria. The master list(s) may be strictly ordered, or may include ties, and the lists of individuals may involve ties and may include all, or just some, of the members of the opposite sex. In fact, ties are almost inevitable in the master list if the ranking is done on the basis of a scoring scheme with a relatively small range of distinct values. We show that many of the interesting variants of stable marriage that are NP-hard remain so under very severe restrictions involving the presence of master lists, but a number of special cases can be solved in polynomial time. Under this master list model, versions of the stable marriage problem that are already solvable in polynomial time typically yield to faster and/or simpler algorithms, giving rise to simple new structural characterisations of the solutions in these cases.

Keywords: Stable matching; Ties; Weakly stable; Strongly stable; Super-stable

1. Introduction and background

The classical stable marriage problem. The Stable Marriage problem (SM) was introduced in the seminal paper of Gale and Shapley [3]. In its classical form, an instance of SM involves n men and n women, each of whom specifies a *preference list*, which is a total order on the members of the opposite sex. A *matching* M is a set of (man, woman) pairs such that each person belongs to exactly one pair. If $(m, w) \in M$, we say that w is m's *partner* in M, and vice versa, and we write $M(m) = w$, $M(w) = m$.

We say that a person x *prefers* y to y' if y precedes y' on x's preference list. A matching M is *stable* if it admits no *blocking pair*, namely a pair (m, w) such that m prefers w to $M(m)$ and w prefers m to $M(w)$. Gale and Shapley [3] proved that every instance of SM admits a stable matching, and described an algorithm – the Gale–Shapley algorithm – that finds such a matching in time that is linear in the input size. In general, there may be many stable matchings (in fact exponentially many in n) for a given instance of SM [13].

Extensions of the classical problem. A variety of extensions of the basic problem have been studied. In the Stable Marriage problem with Incomplete lists (SMI), the numbers of men and women need not be the same, and each person

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p's preference list consists of a subset of the members of the opposite sex (the *acceptable* persons for p) in strict order. A pair (m, w) is *acceptable* if each member of the pair is acceptable to the other. We let a denote the total number of acceptable pairs. A *matching* M is now a set of acceptable pairs such that each person belongs to at most one pair. In this context, (m, w) is a blocking pair for a matching M if (a) (m, w) is an acceptable pair, (b) m is either unmatched or prefers w to $M(m)$, and likewise (c) w is either unmatched or prefers m to $M(w)$. As in the classical case, there is always at least one stable matching for an instance of SMI, and it is straightforward to extend the Gale–Shapley algorithm to this case. Again, there may be many different stable matchings, but Gale and Sotomayor [4] showed that every stable matching for a given SMI instance has the same size and matches exactly the same set of people. We remark that, from the point of view of finding a stable matching, we lose no generality in assuming that, given an instance of SMI, the preference lists are *consistent* (i.e., for any two persons p and q, p is acceptable to q if and only if q is acceptable to p).

The Gale–Shapley algorithm for SM or SMI can be *man-oriented* or *woman-oriented*, i.e., applied from either the men's or the women's 'side'. In the former case, it yields a stable matching – the *man-optimal* – that is simultaneously the best possible stable matching for all of the men and the worst possible for all of the women. The roles of the sexes may be reversed to give the *woman-optimal* stable matching. Some alternative, perhaps fairer, optimality criteria have been proposed. For example, a *minimum regret* stable matching is one for which $\max r(p, M(p))$ (defined as the *regret* of M) is minimised, where the maximum is taken over all persons p, and $r(x, y)$ represents the *rank* of y in the preference list of x. An *egalitarian* stable matching is one for which $\sum r(p, M(p))$ (defined as the *weight* of M) is minimised, where the sum is taken over all persons p. Finally, a *lexicographic maximum* stable matching is one in which the maximum number of people obtain their first-choice partner, and subject to this condition, the maximum number obtain their second-choice partner, and so on. More precisely, for a matching M define $r_i(M)$ to be the number of people for whom $r(p, M(p)) = i$. Then the requirement is a stable matching M for which the vector $(r_1, \ldots, r_n)$ is lexicographically maximum.

Efficient algorithms have been devised for a number of variants of SM and SMI; for example:

- all of the stable pairs (i.e., the (man, woman) pairs that belong to at least one stable matching) can be identified in $O(a)$ time [5];
- all of the stable matchings can be found in $O(a + nk)$ time [5], where k is the total number of such matchings;
- a minimum regret stable matching can be found in $O(a)$ time [5];
- an egalitarian stable matching can be found in $O(a^2)$ time [14], later improved to $O(a^{3/2})$ time [2];
- a lexicographic maximum stable matching can be found in $O(n^{1/2}a^{3/2})$ time [2].

An alternative extension of SM arises if preference lists are allowed to contain *ties*. In the Stable Marriage problem with Ties (SMT) each person's preference list is a partial order over the members of the opposite sex in which indifference is transitive. In other words, each person p's list can be viewed as a sequence of ties, each of length ≥ 1; p prefers each member of a tie to everyone in any subsequent tie, but is indifferent between the members of any single tie. In this context, three definitions of stability have been proposed [6,11].

A matching M is

- *weakly stable* if there is no pair (m, w), each of whom prefers the other to his/her partner in M;
- *strongly stable* if there is no pair (p, q) such that p prefers q to $M(p)$ and q either prefers p to $M(q)$ or is indifferent between them (note that p may be either a man or a woman here);
- *super-stable* if there is no pair (m, w), each of whom either prefers the other to his/her partner in M or is indifferent between them.

It is immediate from the definitions that

$$\text{super-stable} \Rightarrow \text{strongly stable} \Rightarrow \text{weakly stable.}$$

For a given instance of SMT, a weakly stable matching is bound to exist, and can be found in $O(n^2)$ time by breaking all ties in an arbitrary way (i.e., by strictly ranking the members of each tie arbitrarily) and applying the Gale–Shapley algorithm. A super-stable matching may or may not exist, but there is a $O(n^2)$ algorithm to find such a matching or report that there is none [11]. Likewise, a strongly stable matching may or may not exist, but there is a $O(n^4)$ algorithm to find one or report that none exists [11]. An improved $O(n^3)$ version of this latter algorithm has been described recently [18].