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On the two-dimensional orthogonal drawing of series-parallel graphs

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ABSTRACT

It has been known that every planar 4-graph has a 2-bend 2-D orthogonal drawing, with the only exception being the octahedron, every planar 3-graph has a 1-bend 2-D orthogonal drawing with the only exception being K_4, and every outerplanar 3-graph with no triangles has a 0-bend 2-D orthogonal drawing. We show in this paper that every series-parallel 4-graph has a 1-bend 2-D orthogonal drawing.

1. Introduction

We consider the problem of generating orthogonal drawings of graphs in the plane. The problem has obvious applications in the design of VLSI circuits and optoelectronic integrated systems: see for example [7,10].

Throughout this paper, we consider simple connected graphs G with vertex set $V(G)$ and edge set $E(G)$. We denote by $d_G(v)$ the degree of a vertex v in G, and by $\Delta(G)$ the maximum degree of vertices of G. G is called a *k-graph* if $\Delta(G) \leq k$. A graph is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends. Such a drawing of a planar graph G is called a *2-D drawing* of G. A *2-D orthogonal drawing* of a planar graph G is a 2-D drawing of G such that each edge is drawn by a sequence of contiguous horizontal and vertical line segments. Notice that a graph G has a 2-D orthogonal drawing only if $\Delta(G) \leq 4$. A 2-D orthogonal drawing with no more than b bends per edge is called a *b-bend 2-D orthogonal drawing*.

Biedl and Kant [1], and Liu, Morgana, and Simeone [5] showed that every planar 4-graph has a 2-bend 2-D orthogonal drawing, with the only exception being the octahedron shown in Fig. 1(a), which has a 3-bend 2-D orthogonal drawing, as shown in Fig. 1(b). Moreover, Kant [4] showed that every planar 3-graph has a 1-bend 2-D orthogonal drawing with the only exception being K_4 shown in Fig. 1(c), which has a 2-bend 2-D orthogonal drawing, as shown in Fig. 1(d). Zhou and Nishizeki [11] showed a linear time algorithm to generate a 1-bend 2-D orthogonal drawing for a series-parallel 3-graph. Nomura, Tayu, and Ueno [6] showed that every outerplanar 3-graph has a 0-bend 2-D orthogonal drawing if and only if it contains no triangle as a subgraph. On the other hand, Garg and Tamassia proved that it is $\mathcal{NP}$-complete to decide if a given planar 4-graph has a 0-bend 2-D orthogonal drawing [3]. Di Battista, Liotta, and Vargiu showed that the problem can be solved in polynomial time for planar 3-graphs and series-parallel graphs [2]. Rahman, Egi, and Nishizeki [8] showed that the problem can be solved in linear time for series-parallel 3-graphs.

We show in this paper the following theorem.

Theorem 1. *Every series-parallel 4-graph has a 1-bend 2-D orthogonal drawing.* □

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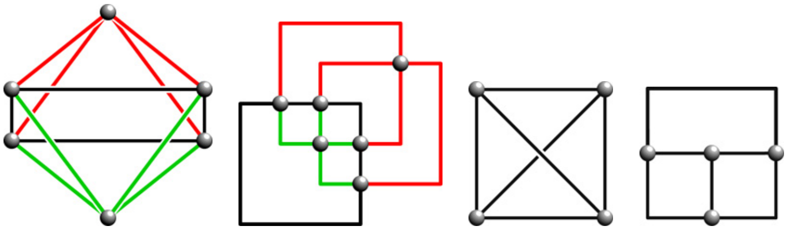
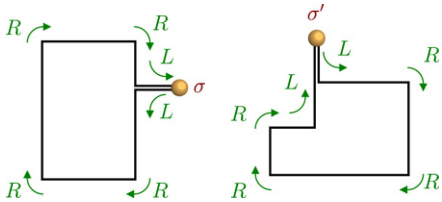


(a) Octahedron. (b) 3-bend 2-D orthogonal drawing of octahedron. (c) K_4. (d) 2-bend 2-D orthogonal drawing of K_4.

Fig. 1. Octahedron, K_4, and their 2-D orthogonal drawings.



(a) Λ_1. (b) Λ'_1.

Fig. 2. Shape-equivalent polygons Λ_1 and Λ'_1.

The proof of Theorem 1 is constructive and provides a polynomial-time algorithm to generate such a drawing for a series-parallel 4-graph.

2. Preliminaries

A *series-parallel graph* is defined recursively as follows.

(1) A graph consisting of two vertices joined by a single edge is a series-parallel graph. The vertices are the *terminals*.
(2) If G_1 is a series-parallel graph with terminals s_1 and t_1, and G_2 is a series-parallel graph with terminals s_2 and t_2, then a graph G obtained by either of the following operations is also a series-parallel graph:
  (i) *Series-composition*: identify t_1 with s_2. Vertices s_1 and t_2 are the terminals of G.
  (ii) *Parallel-composition*: identify s_1 and s_2 into a vertex s, and t_1 and t_2 into t. Vertices s and t are the terminals of G.

A series-parallel graph G is naturally associated with a binary tree $T(G)$, which is called a *decomposition tree* of G. The nodes of $T(G)$ are of three types, S-nodes, P-nodes, and Q-nodes. $T(G)$ is defined recursively as follows:

(1) If G is a single edge, then $T(G)$ consists of a single Q-node.
(2-i) If G is obtained from series-parallel graphs G_1 and G_2 by the series-composition, then the root of $T(G)$ is an S-node, and $T(G)$ has subtrees $T(G_1)$ and $T(G_2)$ rooted at the children of the root of G.
(2-ii) If G is obtained from series-parallel graphs G_1 and G_2 by the parallel-composition, then the root of $T(G)$ is a P-node, and $T(G)$ has subtrees $T(G_1)$ and $T(G_2)$ rooted at the children of the root of G.

Notice that the leaves of $T(G)$ are the Q-nodes, and an internal node of $T(G)$ is either an S-node or P-node. Notice, also, that every P-node has at most one Q-node as a child, since G is a simple graph. If G has n vertices then $T(G)$ has $\mathcal{O}(n)$ nodes, and $T(G)$ can be constructed in $\mathcal{O}(n)$ time [9]. It should be noted that the decomposition tree defined here is slightly different from the well-known SPQ-tree for a series-parallel graph.

A polygon is said to be *rectilinear* if every edge of the polygon is parallel to the horizontal or the vertical axes. Let Λ and Λ' be rectilinear polygons with distinguished vertices σ and σ', respectively. Λ and Λ' are said to be *shape-equivalent* if walking clockwise around Λ and Λ' from σ and σ', respectively, we have the same sequence of left and right turns for Λ and Λ'. Fig. 2 shows shape-equivalent rectilinear polygons Λ_1 and Λ'_1 whose corresponding sequence is (L, R, R, R, R, L), where L and R denote left and right turns, respectively.

Let Λ be a rectilinear polygon with distinguished vertices σ and τ, and Λ' be a rectilinear polygon with distinguished vertices σ' and τ'. Λ and Λ' are *shape-equivalent* if walking clockwise around Λ and Λ' from σ and σ', respectively, we have the same sequence of left turns, right turns, and the direction (left turn, right turn, or go straight) at τ and τ' for Λ and Λ', respectively. Fig. 3 shows shape-equivalent rectilinear polygons Λ_2 and Λ'_2 whose corresponding sequence is $S = (L, R, R, R, L, F, L, R, R, R, L)$, where F denotes the direction of going straight at τ and τ'. On the other hand, a rectilinear polygon shown in Fig. 4 is not shape-equivalent to Λ_2 or Λ'_2, since the sequence $(L, R, R, R, L, R^{\dagger}, L, R, R, R, L)$ is different from S, where $R^{\dagger}$ denotes the right turn at τ''.

Any two rectilinear rectangles with no distinguished vertex are defined to be *shape-equivalent*.