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Number-theoretic interpretation and construction of a digital circle

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Abstract

This paper presents a new interpretation of a digital circle in terms of the distribution of square numbers in discrete intervals. The number-theoretic analysis that leads to many important properties of a digital circle succinctly captures the original perspectives of digital calculus and digital geometry for its visualization and characterization. To demonstrate the capability and efficacy of the proposed method, two simple algorithms for the construction of digital circles, based on simple number-theoretic concepts, have been reported. Both the algorithms require only a few primitive operations and are completely devoid of any floating-point computation. To speed up the computation, especially for circular arcs of high radii, a hybridized version of these two algorithms has been given. Experimental results have been furnished to elucidate the analytical power and algorithmic efficiency of the proposed approach. It has been also shown, how and why, for sufficiently high radius, the number-theoretic technique can expedite a circle construction algorithm.

Keywords: Digital circle; Digital geometry; Number theory; Bresenham's algorithm; Computer graphics

1. Introduction

Characterization and construction of the simple yet prevalent geometric primitives, especially straight lines and circles, in the discrete domain, hail long back from their successful realization in graphics display [1,20]. Today, with the evolution of new in-theory digital paradigms, such as digital calculus [36], digital geometry [28], theory of words and numbers in the digital perspective [29,34], etc., the analytical studies and theoretical interpretations of these geometric primitives have become inevitable in order to augment different inter-theoretical relations and applications, and to actuate and observe the comeback of classical theories into the formation and augmentation of emerging theories.

Digital circles, abiding with their weird and challenging nature in the discrete domain, have drawn immense research interest since the early adoption of the scan-conversion technique [2,6,12,14,17,24,30,38,42] for their efficient approximation from the continuous domain to the digital domain. Subsequent improvements of these algorithms meant for generation of circular arcs were achieved by different researchers in the later periods, which may be seen in [5,7,26,33,43,46–48]. Further, apart from the circle-generation algorithms, since the properties, parameterization, characterization, and recognition of digital circles and circular arcs shape up a very engrossing area of research, several other interesting works on digital circles and related problems have also appeared from time to time, some of which are as follows:

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Table 1
A comparative study of some existing approaches with the proposed one

Algorithm	Features	Scan conversion[a]	Readiness for hybridization	Interval searching[b]
1. Incremental algorithm [6]	Decision function based on minimum-residual criterion	Yes	No	No
2. Optimized midpoint algorithm ([20], *DCB*: Fig. 7)	First- and second-order differences to expedite the computation of decision function	Yes	No	No
3. Short run algorithm [26]	Selection of minimum-distance pixels	No	No	No
4. Hybrid run length slice algorithm [48][c]	(i) uses run length properties (ii) uses decision variable (d)[d]	No[e]	Yes	No
5. Number-theoretic algorithm [proposed]	(i) uses run length properties (ii) run lengths are binary searched in integer intervals using the predicted upper and lower bounds of runs	No	Yes	Yes

[a] pixels are generated one in each iteration in accordance with the decision function derived from some chosen criterion.
[b] instead of pixels, runs are generated by searching the largest square number in an integer interval.
[c] an improvement and hybridization of the algorithm by [26].
[d] computation of d is aided by double-step and quadruple-step forward differencing for non-singular runs.
[e] scans the "runs of pixels" instead of scanning the "pixels of runs".

– *polygonal approximation of digital circles*: [3,25];
– *characterization of digital circles*: [4,18,22,35,45];
– *detection/segmentation of circular arcs/objects in digital images*: [9–11,13,15,23,27,31,36,39,40];
– *parameterization of circular arcs*: [8,16,44,49];
– *anti-aliasing solutions for digital circles*: [19]; etc.

It may be mentioned that there exist various composite techniques [5,26,46–48], designed to expedite and contend the procedure of Bresenham's Circle Drawing algorithm [6]. However, these algorithms do not have considerably large gains over Bresenham's method. In fact, Bresenham's algorithm cannot be overtaken by a great margin by some other algorithm, which has been verified by the other algorithms developed in later years. For instance, for radii from 1 to 128, the most efficient ones (please see, for example, the algorithms and results by Hsu et al. [26], and by Yao and Roken [48]) of these algorithms are 0.62 times to 1.53 times faster on the average with respect to Bresenham's algorithm. This is quite expected for the inherent brevity and simplicity of Bresenham's algorithm owing to the judicious usage of primitive arithmetic operations, which have been worked out sensibly and skillfully from the fundamentals of digital calculus.

However, the essence of all these algorithms is that, although a circle drawing algorithm primarily originates from the naive concept of intelligent digital applications of first-order and second-order derivatives (differences) as in Bresenham's, a digital circle is endowed with some other interesting properties contributed by the classical theories in the discrete domain. It has been gradually made apparent from these works that digital circles, similar to digital straight lines [29], possess some striking characteristics, which, if interpreted rightly and exploited properly, may produce interesting results and scope for subsequent potential applications in the discrete domain.

This paper reveals the relation between perfect squares (square numbers) in discrete (integer) intervals and few interesting and useful properties of a digital circle. Based on these number-theoretic properties, the problem of constructing a digital circle or a circular arc maps to the new domain of number theory. However, the construction of a digital circle using these properties should not be treated only in terms of its performance compared to Bresenham's algorithm or any other similar algorithm; rather, these number-theoretic properties enrich the understanding of digital circles from a perspective that is different from the customary aspects of digital calculus. It is also discussed in this paper how these intervals can be obtained for the given radius of a circle, and what effects these intervals have on the construction of a digital circle. For a brief overview, in Table 1, we have presented a comparison of our method with some existing methods.