Discrete Applied Mathematics 157 (2009) 1229–1241

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

0–1 reformulations of the multicommodity capacitated network design problem

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ARTICLE INFO

Article history:
Received 15 January 2007
Received in revised form 20 August 2007
Accepted 14 April 2008
Available online 2 June 2008

Keywords:
Multicommodity capacitated network design
Reformulation
Valid inequalities
Cutting-plane method
Column generation

ABSTRACT

We study 0–1 reformulations of the multicommodity capacitated network design problem, which is usually modeled with general integer variables to represent design decisions on the number of facilities to install on each arc of the network. The reformulations are based on the multiple choice model, a generic approach to represent piecewise linear costs using 0–1 variables. This model is improved by the addition of extended linking inequalities, derived from variable disaggregation techniques. We show that these extended linking inequalities for the 0–1 model are equivalent to the residual capacity inequalities, a class of valid inequalities derived for the model with general integer variables. In this paper, we compare two cutting-plane algorithms to compute the same lower bound on the optimal value of the problem: one based on the generation of residual capacity inequalities within the model with general integer variables, and the other based on the addition of extended linking inequalities to the 0–1 reformulation. To further improve the computational results of the latter approach, we develop a *column-and-row generation* approach; the resulting algorithm is shown to be competitive with the approach relying on residual capacity inequalities.

1. Introduction

The *multicommodity capacitated network design problem* (MCND) we consider is defined on a directed network $G = (N, A)$, where N is the set of nodes and A is the set of arcs. We must satisfy the communication demands between several origin-destination pairs, represented by the set of commodities K. For each commodity k, we denote by d^k the positive demand that must flow between the origin $O(k)$ and the destination $D(k)$. While flowing along an arc (i, j), a communication consumes some of the arc capacity; the capacity is obtained by installing on some of the arcs any number of *facilities* of a single type. Installing one facility on arc (i, j) provides a positive capacity u_{ij} at a (nonnegative) cost f_{ij}. A nonnegative routing cost c_{ij}^k also has to be paid for each unit of commodity k moving through arc $(i, j) \in A$. The problem consists of minimizing the sum of all costs, while satisfying demand requirements and capacity constraints. Several applications in transportation, logistics, telecommunications, and production planning can be represented as variants of this classical network design problem [7,22,34,47,48].

Defining nonnegative *flow variables* x_{ij}^k, which represent the fraction of the flow of commodity k on arc $(i, j) \in A$ (i.e., $d^k x_{ij}^k$ is the flow of commodity k on arc (i, j)) and general integer *design variables* y_{ij}, which define the number of facilities to install on

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0166-218X/$ – see front matter
doi:10.1016/j.dam.2008.04.022

arc (i,j), the problem can then be formulated as the following mixed-integer programming (MIP) model, which we denote I,

$$\min \sum_{k\in K}\sum_{(i,j)\in A} d^k c_{ij}^k x_{ij}^k + \sum_{(i,j)\in A} f_{ij} y_{ij} \quad (1)$$

$$\sum_{j\in N_i^+} x_{ij}^k - \sum_{j\in N_i^-} x_{ji}^k = \delta_i^k \quad i \in N,\ k \in K \quad (2)$$

$$\sum_{k\in K} d^k x_{ij}^k \le u_{ij} y_{ij} \quad (i,j) \in A \quad (3)$$

$$0 \le x_{ij}^k \le 1 \quad (i,j) \in A,\ k \in K \quad (4)$$

$$y_{ij} \ge 0 \quad (i,j) \in A \quad (5)$$

$$y_{ij} \text{ integer}, \quad (i,j) \in A \quad (6)$$

where $N_i^+ = \{j \in N | (i,j) \in A\}$, $N_i^- = \{j \in N | (j,i) \in A\}$, and $\delta_i^k = 1$ if $i = O(k)$, $\delta_i^k = -1$ if $i = D(k)$, and $\delta_i^k = 0$ otherwise.

Although MCND may appear to be a crude approximation of the complex network design problems that arise in practice, it is a flexible modeling tool that can be adapted to a large number of application areas. In particular, various extensions have been studied, most notably in transportation [11,29,38,43] and telecommunications [12,16,18]. One of these extensions is the *unsplittable* (or nonbifurcated) variant of the problem, where the flow of each commodity is required to follow one route between the origin and the destination. These constraints require the flow variables to be binary, substantially increasing the difficulty of the problem. Another extension is the *capacity expansion* variant of the problem, where some of the arcs already have an existing capacity. In this case, the network has to be expanded, rather than designed from scratch.

An important special case of MCND is the *network loading problem* (NL), where a limited number (generally, one or two) of facility types, each with given unit cost and capacity, are available on all arcs. Several algorithms have been proposed to solve NL, in particular heuristic approaches to solve large-scale instances (with more than two facility types) [1,14,35] and branch-and-cut (B&C) methods based on polyhedral analysis [2,9,15,17,19,39,44–46,49]. These B&C algorithms all make use of the so-called *cutset inequalities*, which state that there should be enough capacity installed to satisfy the demands that must be routed across any cutset. Extensions of these inequalities have also been studied, in particular the *metric inequalities*, which specify necessary and sufficient conditions for the feasibility of any multicommodity capacitated network flow problem [21,32,33]. The latter are especially interesting, as when there are no routing costs, they can be used to project out the flow variables, thus simplifying the model.

Another interesting variant of MCND occurs when we restrict the design variables to 0–1 values, thus obtaining the *fixed-charge* MCND (FMCND). Research on this problem has focused on heuristic methods [24,25,36,37], bounding procedures and exact approaches [20,23,34,40,42,53]. Some conclusions from these studies are useful as well for the MCND. In particular, it is noteworthy that the linear programming (LP) relaxation of FCMND provides an extremely weak approximation of the MIP model. However, by appending the following simple forcing constraints:

$$x_{ij}^k \le y_{ij}, \quad (i,j) \in A,\ k \in K, \quad (7)$$

called *strong linking inequalities*, we obtain a significantly improved LP lower bound. Also, bounding methods based on the Lagrangian relaxation of constraints (2) have been shown to perform well in practice. It is interesting to note that the convex hull of solutions to the resulting Lagrangian subproblem is obtained by adding the strong linking inequalities (7).

These inequalities are obviously valid for the MCND, but they are weak in this case, as they are not forcing constraints. The *residual capacity inequalities*, introduced by Magnanti, Mirchandani and Vachani [45] and later studied by Atamtürk and Rajan [4] and Atamtürk and Günlük [3], generalize the strong linking inequalities, in the sense that they describe the convex hull of solutions to the subproblem obtained from the Lagrangian relaxation of (2). An alternative to compute the same bound as the Lagrangian dual resulting from this relaxation is to append the residual capacity inequalities to the model, obtaining formulation I^+, and solve the LP relaxation of I^+. One difficulty with this approach is the *exponential* number of residual capacity inequalities. Hence, researchers have used residual capacity inequalities within iterative cutting-plane methods [4,46]. The success of such approaches depends on the ability to solve the *separation problem*: given a solution to the current LP relaxation, is it possible to find efficiently violated residual capacity inequalities? Atamtürk and Rajan [4] provide a positive answer to this question. We recall this result in Section 2, which reviews the state-of-the-art on the residual capacity inequalities.

An alternative to I, the model with general integer design variables, is to view MCND as a nonlinear nonconvex minimization problem. In particular, the design costs can be modeled using piecewise linear staircase functions, one for each arc, the number of facilities installed on the arc corresponding to a "step", or *segment*, of this function. One can attempt to solve directly the resulting nonlinear model [52], or first reformulate it as a 0–1 MIP model [51], using any of the classical techniques to model piecewise linear functions with 0–1 variables [26,41]. One of these 0–1 reformulations, the *multiple choice model*, hereafter denoted B, has been studied in the context of multicommodity network flow problems [6,27,28]. These studies show that, by defining additional flow variables, one can append to the LP relaxation a class of forcing constraints called the *extended linking inequalities* to obtain high-quality lower bounds, at the