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Spanning forests and the golden ratio

Pavel Chebotarev

Trapeznikov Institute of Control Sciences of the Russian Academy of Sciences, 65 Profsoyuznaya Street, Moscow 117997, Russia

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Abstract

For a graph G, let f_{ij} be the number of spanning rooted forests in which vertex j belongs to a tree rooted at i. In this paper, we show that for a path, the f_{ij}'s can be expressed as the products of Fibonacci numbers; for a cycle, they are products of Fibonacci and Lucas numbers. The *doubly stochastic graph matrix* is the matrix $F=(f_{ij})_{n\times n}/f$, where f is the total number of spanning rooted forests of G and n is the number of vertices in G. F provides a proximity measure for graph vertices. By the matrix forest theorem, $F^{-1}=I+L$, where L is the Laplacian matrix of G. We show that for the paths and the so-called T-caterpillars, some diagonal entries of F (which provide a measure of the self-connectivity of vertices) converge to ϕ^{-1} or to $1-\phi^{-1}$, where ϕ is the golden ratio, as the number of vertices goes to infinity. Thereby, in the asymptotic, the corresponding vertices can be metaphorically considered as "golden introverts" and "golden extroverts," respectively. This metaphor is reinforced by a Markov chain interpretation of the doubly stochastic graph matrix, according to which F equals the overall transition matrix of a random walk with a random number of steps on G.

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Keywords: Doubly stochastic graph matrix; Matrix forest theorem; Fibonacci numbers; Laplacian matrix; Vertex–vertex proximity; Spanning forest; Golden ratio

1. Introduction

Let $G=(V,E)$ be a simple graph with vertex set $V=V(G)$, $|V|=n$, and edge set $E=E(G)$. Suppose that $n\geqslant 2$.

A *spanning rooted forest of* G is any spanning acyclic subgraph of G with a single vertex (a *root*) marked in each tree.

Let $f_{ij}=f_{ij}(G)$ be the number of spanning rooted forests of G in which vertices i and j belong to the same tree rooted at i. The matrix $(f_{ij})_{n\times n}$ is the *matrix of spanning rooted forests of* G. Let $f=f(G)$ be the total number of spanning rooted forests of G.

The matrix $F=(f_{ij})_{n\times n}/f$ is referred to as the *doubly stochastic graph matrix* [14,15,24,23] or the *matrix of relative connectivity via forests*. By the matrix forest theorem [7,8,5],

$$F^{-1}=I+L \tag{1}$$

E-mail addresses: chv@member.ams.org, upi@ipu.ru.

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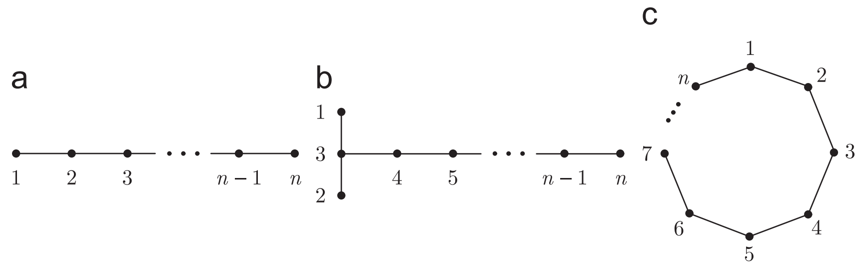

Fig. 1. (a) the path P_n; (b) the T-caterpillar T_n; and (c) the cycle C_n.

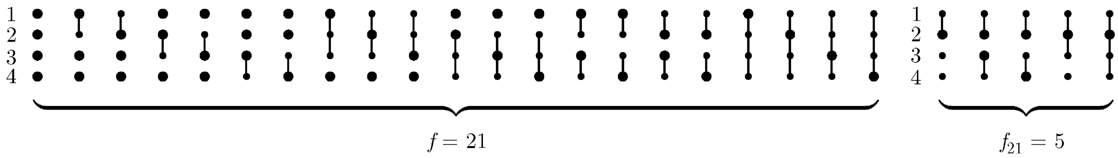

Fig. 2. The spanning rooted forests in P_4 and the forests where 1 is in a tree rooted at 2.

and

$$f = \det(I + L), \tag{2}$$

where L is the *Laplacian matrix* of G, i.e. $L = D - A$, A being the adjacency matrix of G and D the diagonal matrix of vertex degrees of G. Most likely, the matrix $(I + L)^{-1} = F$ was first considered in [11]. Chaiken [4] used the matrix $\operatorname{adj}(I + L) = (f_{ij})_{n\times n}$ for coordinatizing linking systems of strict gammoids. The (i, j) entry of F can be considered as a measure of proximity between vertices i and j in G; the (i, i) entry measures the self-connectivity of vertex i.

A *path* is a connected graph in which two vertices have degree 1 and the remaining vertices have degree 2. Let P_n be the path with $V(P_n) = \{1, 2, \dots, n\}$ and $E(P_n) = \{(1, 2), (2, 3), \dots, (n - 1, n)\}$, see Fig. 1(a). All spanning rooted forests of P_4 and the spanning rooted forests in which vertex 1 belongs to a tree rooted at vertex 2 are shown in Fig. 2, where thick dots denote roots. The matrix F for P_4 is

$$F(P_4) = \frac{(f_{ij})}{f} = \frac{1}{21}\begin{bmatrix} 13 & 5 & 2 & 1 \\ 5 & 10 & 4 & 2 \\ 2 & 4 & 10 & 5 \\ 1 & 2 & 5 & 13 \end{bmatrix}.$$

Let T_n be the graph obtained from P_n by replacing the edge $(1, 2)$ with $(1, 3)$: $V(T_n) = \{1, 2, \dots, n\}$ and $E(T_n) = \{(1, 3), (2, 3), (3, 4), \dots, (n - 1, n)\}$, see Fig. 1(b). We call T_n a *T-caterpillar*.

Let C_n, $n \geqslant 3$, be the *cycle* on n vertices: $V(C_n) = \{1, 2, \dots, n\}$ and $E(C_n) = \{(1, 2), (2, 3), \dots, (n - 1, n), (n, 1)\}$, Fig. 1(c).

By $(\Phi_i)_{i=0,1,2,\dots} = (0, 1, 1, 2, 3, 5, \dots)$ we denote the Fibonacci numbers. Sometimes, it is convenient to consider the subsequences of Fibonacci numbers with odd and even subscripts separately:

$$\Phi'_i = \Phi_{2i-1}, \quad i = 1, 2, \dots,$$
$$\Phi''_i = \Phi_{2i}, \quad i = 0, 1, 2, \dots .$$

In Section 2 we study the spanning rooted forests in paths, cycles, and T-caterpillars, in Section 3 the results are interpreted in terms of vertex–vertex proximity, and Sections 4 and 5 present interpretations of the doubly stochastic graph matrix in terms of random walks and information dissemination, respectively.

2. Spanning rooted forests in paths, cycles, and T-caterpillars

Theorem 1. *Let G be a path, $G = P_n$. Then $f = \Phi''_n$ and $f_{ij} = \Phi'_{\min(i,j)} \cdot \Phi'_{n+1-\max(i,j)}$ for all $i, j = 1, \dots, n$.*