Discrete Applied Mathematics 157 (2009) 1113–1127

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Average distances and distance domination numbers☆

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ARTICLE INFO

Article history:
Received 28 May 2005
Received in revised form 27 February 2008
Accepted 9 March 2008
Available online 25 April 2008

Keywords:
Distance
Average distance
Diameter
Domination number
Distance domination number

ABSTRACT

Let k be a positive integer and G be a simple connected graph with order n. The average distance $\mu(G)$ of G is defined to be the average value of distances over all pairs of vertices of G. A subset D of vertices in G is said to be a k-dominating set of G if every vertex of $V(G)-D$ is within distance k from some vertex of D. The minimum cardinality among all k-dominating sets of G is called the k-domination number $\gamma_k(G)$ of G. In this paper tight upper bounds are established for $\mu(G)$, as functions of n, k and $\gamma_k(G)$, which generalizes the earlier results of Dankelmann [P. Dankelmann, Average distance and domination number, Discrete Appl. Math. 80 (1997) 21–35] for $k=1$.

1. Introduction

For terminology and notation on graph theory not given here, the reader is referred to [18]. Let $G=(V,E)$ be a finite simple connected graph with vertex set $V=V(G)$ and edge set $E=E(G)$. The *distance* $d_G(x,y)$ between two vertices x and y is the length of a shortest xy-path in G. For $S\subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S and for $v\in V(G)$, $d_G(v,S)=\min\{d_G(v,u):\ u\in V(S)\}$. The *eccentricity* $e_G(v)$ of v is $\max\{d_G(v,x):\ x\in V(G)\}$. The *radius* $\mathrm{rad}(G)$ and the *diameter* $\mathrm{diam}(G)$ of G are the smallest and the largest eccentricities of the vertices in G, respectively. A vertex with $e_G(v)=\mathrm{diam}(G)$ is called a *diametral vertex*. A vertex v is a *central vertex* if $e_G(v)=\mathrm{rad}(G)$ and the *center* of G is the set of all central vertices. The *degree* of a vertex $x\in V(G)$, denoted by $\deg_G(x)$, is the number of edges incident to the vertex x. A vertex of degree one is called an *end-vertex*. Let P_n denote a path of order n and P_{xy} a path with end-vertices x and y. If the length of a path P_{xy} is equal to $\mathrm{diam}(G)$, then we call P_{xy} a *diametral path* in G.

The average (or mean) distance of G is defined to be the average over all pairs of vertices of G, i.e.,

$$\mu(G)=\frac{1}{n(n-1)}\sum_{x,y\in V}d_G(x,y).$$

Like diameter, Wiener index [13,17] or other parameters, apart from their own graph-theoretic interests, the average distance has numerous applications in analyzing problems in communication networks, geometry and physical chemistry. It is the reason why this concept has received considerable attention in the literature. There are several excellent surveys of earlier results on average distance of graphs, one of which is due to Plesnik [15]. Thus, many efforts have been made by several authors to establish the relationships between average distance and other graph parameters (see, for example, [1,2,

☆ The work was supported by NNSF of China (No. 10671191).

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doi:10.1016/j.dam.2008.03.024

6–8,15,16]). For convenience, let

$$\sigma(x) = \sigma(x, G) = \sum_{y \in V} d_G(x, y), \qquad \sigma(G) = \sum_{x \in V} \sigma(x) = \sum_{(x,y) \in V \times V} d_G(x, y),$$

be the *transmission* of a vertex $x \in V$, and the *transmission* of the graph G, respectively. In order to avoid large fractions, we will often deal with $\sigma(G)$ rather than $\mu(G)$. Apart from average distance, $\sigma(G)$ also occurs in the computation of other graph-theoretical parameters, such as the forwarding index of a routing [5,12], and physical chemistry [9].

A subset I of vertices in G is said to be *k-independent* if every vertex in I is at distance at least $k + 1$ from every other vertex of I in G. The *k-independence number* of G, denoted by $\alpha_k(G)$, is defined to be the maximum cardinality among all k-independent sets of G. If $k = 1$, $\alpha_1(G)$ is $\alpha(G)$, the *independence number* of G. Dankelmann, Oellermann and Swart [7] gave the bounds on the average distance with order n and independence number $\alpha(G)$. Firby and Haviland [8] established sharp lower bounds for the average distance of G, in terms of the k-independence number $\alpha_k(G)$, and described the associated extremal graphs, thereby extending the aforementioned work of Dankelmann et al. for $k = 1$.

A subset D of vertices in G is said to be a *k-dominating set* of G if every vertex of $V(G) - D$ is within distance k from some vertex of D. The minimum cardinality among all k-dominating sets of G is called the *k-domination number* of G and is denoted by $\gamma_k(G)$. For the special case of $k = 1$, $\gamma_1(G)$ is the classic *domination number* of G. The concept of k-dominating set was introduced by Chang and Nemhauser [3,4] and finds applications in many situations and structures which give rise to graphs, see the books by Haynes, Hedetniemi and Slater [10,11].

Dankelmann [6] gave the sharp upper bounds on the average distance of a graph of given order n and domination number $\gamma(G)$, and determined the extremal graphs. In this paper, by generalizing Dankelmann's technique, we establish the sharp upper bounds on the average distance of G, in terms of k-domination number $\gamma_k(G)$, and describe the extremal graphs, extending the results of Dankelmann for $k = 1$ in [6].

The proofs of our main results are in Section 3 and some lemmas are given in Section 2.

2. Lemmas

Lemma 2.1. *Let G be a nontrivial connected graph, and k be a positive integer. Then $\gamma_k(G) = \min \gamma_k(T)$, where the minimum is taken over all spanning trees T of G.*

Proof. Let G be a nontrivial connected graph and T be a spanning tree of G. Then any k-dominating set of T is also a k-dominating set of G. Therefore, $\gamma_k(G) \le \gamma_k(T)$. Thus we have that $\gamma_k(G) \le \min \gamma_k(T)$, where the minimum is taken over all spanning trees T of G.

Now we show the reverse inequality. If G is a tree, then the theorem holds trivially. So we may assume that G is a connected graph containing cycles. Let D be a minimum k-dominating set of G and C be a cycle in G. If we can prove that D is also a k-dominating set of $G - e$ for some cycle edge $e \in E(C)$, then $\gamma_k(G - e) \le |D| = \gamma_k(G)$. By iterating the above operation finitely, we get $\gamma_k(T) \le \gamma_k(G)$ for some spanning tree T of G. Thus, we have that $\min \gamma_k(T) \le \gamma_k(G)$, where the minimum is taken over all spanning trees T of G.

If $V(C) \subseteq V(D)$, then obviously the vertices in $V(G) - D$ are also all within distance k to $G[D] - e$ for any edge $e \in E(C)$.

If $V(C) \not\subseteq V(D)$, then we select two adjacent vertices x and y in $V(C)$ such that $d_G(x, D) + d_G(y, D) = \max\{d_G(u, D) + d_G(v, D) : uv \in E(C)\}$. Now we will show that D is also a k-dominating set of $G - \{xy\}$.

First for any two adjacent vertices u and v in G, we have $|d_G(u, D) - d_G(v, D)| \le 1$. Then if w is a vertex in $V(C)$ such that $d_G(w, D) = \max\{d_G(v, D) : v \in V(C)\}$, we have that $w = x$ or $w = y$. Without loss of generality, suppose that $d_G(x, D) = \max\{d_G(v, D) : v \in V(C)\}$.

Let z be another neighbor of x different from y in $V(C)$. So we immediately have that $d_G(z, D) \le d_G(y, D)$. Thus, we get the distance between a vertex in $V(G) - D$ and D is not influenced by deleting the edge $\{xy\}$. That is to say, $d_{G-xy}(v, D) = d_G(v, D)$ for all vertices v in $V(G)$. Hence, D is also a k-dominating set of $G - e$ for some cycle edge e. ■

From Lemma 2.1, we get that every connected graph G contains a spanning tree T with the same k-domination number. That is to say, every extremal graph G with given order, k-domination number and maximum average distance is a tree. So we have to consider only trees below.

Let $S(k)$ denote a *k-generalized star* which is a tree containing one vertex whose eccentricity is at most k.

Lemma 2.2. *Let H be a graph. Then $\gamma_k(H - e) > \gamma_k(H)$ for each edge $e \in E(H)$ if and only if H is the union of several vertex disjoint k-generalized stars $S(k)$.*

Proof. Let H be a graph such that $\gamma_k(H - e) > \gamma_k(H)$ for each edge $e \in E(H)$, and D be a minimum k-dominating set of H.

If $\gamma_k(H) = 1$, by Lemma 2.1 and the property $\gamma_k(H - e) > \gamma_k(H)$ for each edge $e \in E(H)$, then H must be a tree and we can easily see that H must be a k-generalized star $S(k)$. If $\gamma_k(H) \ge 2$, then for any two vertices x and y in D, we have $d_H(x, y) \ge 2k + 1$. Otherwise, if $d_H(x, y) \le 2k$, then there must exist an edge e on the shortest path between x and y in H such that $\gamma_k(H - e) = \gamma_k(H)$.

We partition the graph H into balls of radius k, denoted $H_1, H_2, \ldots, H_{\gamma_k}$, whose centers are the vertices in D.