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On the global offensive alliance number of a graph[☆]

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Abstract

An offensive alliance in a graph $\Gamma = (V, E)$ is a set of vertices $S \subset V$ where for each vertex v in its boundary the majority of vertices in v's closed neighborhood are in S. In the case of strong offensive alliance, strict majority is required. An alliance S is called global if it affects every vertex in $V \setminus S$, that is, S is a dominating set of Γ. The *global offensive alliance number* $\gamma_o(\Gamma)$ is the minimum cardinality of a global offensive alliance in Γ. An offensive alliance is connected if its induced subgraph is connected. The *global-connected offensive alliance number*, $\gamma_{co}(\Gamma)$, is the minimum cardinality of a global-connected offensive alliance in Γ.

In this paper we obtain several tight bounds on $\gamma_o(\Gamma)$ and $\gamma_{co}(\Gamma)$ in terms of several parameters of Γ. The case of strong alliances is studied by analogy.

Keywords: Offensive alliances; Global alliances; Domination; Independence number

1. Introduction

The study of defensive alliances in graphs, together with a variety of other kinds of alliances, was introduced in [6]. In the cited paper there was initiated the study of the mathematical properties of alliances. In particular, several bounds on the defensive alliance number were given. The particular case of global (strong) defensive alliance was investigated in [4].

The study of offensive alliances was initiated by Favaron et al. in [2] where there were derived several bounds on the offensive alliance number and the strong offensive alliance number. On the other hand, in [7] there were obtained several tight bounds on different types of alliance numbers of a graph: (global) defensive alliance number, global offensive alliance number and global dual alliance number. In particular, there was investigated the relationship between the alliance numbers of a graph and its algebraic connectivity, its spectral radius, and its Laplacian spectral radius. A particular study of the alliance numbers, for the case of planar graphs, can be found in [9]. Moreover, for the study of defensive alliances in the line graph of a simple graph we cite [11].

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The aim of this paper is to study mathematical properties of the global offensive alliance number and the global strong offensive alliance number of a graph. We begin by stating some notation and terminology. In this paper $\Gamma = (V, E)$ denotes a simple graph of order n and size m. The degree of a vertex $v \in V$ will be denoted by $\delta(v)$, the minimum degree will be denoted by δ, and the maximum degree by Δ. The subgraph induced by a set $S \subset V$ will be denoted by $\langle S \rangle$. For a non-empty subset $S \subset V$, and a vertex $v \in V$, we denote by $N_S(v)$ the set of neighbors that v has in S: $N_S(v) := \{u \in S : u \sim v\}$. Similarly, we denote by $N_{V \setminus S}(v)$ the set of neighbors that v has in $V \setminus S$: $N_{V \setminus S}(v) := \{u \in V \setminus S : u \sim v\}$. The boundary of a set $S \subset V$ is defined as $\partial(S) := \bigcup_{v \in S} N_{V \setminus S}(v)$.

A non-empty set of vertices $S \subset V$ is called an *offensive alliance* if and only if for every $v \in \partial(S)$, $|N_S(v)| \geq |N_{V \setminus S}(v)| + 1$. That is, a non-empty set of vertices $S \subset V$ is called an offensive alliance if and only if for every $v \in \partial(S)$, $2|N_S(v)| \geq \delta(v) + 1$.

An offensive alliance S is called *strong* if for every vertex $v \in \partial(S)$, $|N_S(v)| \geq |N_{V \setminus S}(v)| + 2$. In other words, an offensive alliance S is called strong if for every vertex $v \in \partial(S)$, $2|N_S(v)| \geq \delta(v) + 2$.

The *offensive alliance number* (respectively, *strong offensive alliance number*), denoted as $a_o(\Gamma)$ (respectively, $a_{\hat{o}}(\Gamma)$), is defined as the minimum cardinality of an offensive alliance (respectively, a strong offensive alliance) in Γ.

A non-empty set of vertices $S \subset V$ is a *global offensive alliance* if for every vertex $v \in V \setminus S$, $|N_S(v)| \geq |N_{V \setminus S}(v)| + 1$. Thus, global offensive alliances are also dominating sets, and one can define the *global offensive alliance number*, denoted as $\gamma_o(\Gamma)$, to equal the minimum cardinality of a global offensive alliance in Γ. Analogously, $S \subset V$ is a *global strong offensive alliance* if for every vertex $v \in V \setminus S$, $|N_S(v)| \geq |N_{V \setminus S}(v)| + 2$, and the *global strong offensive alliance number*, denoted as $\gamma_{\hat{o}}(\Gamma)$, is defined as the minimum cardinality of a global strong offensive alliance in Γ.

It was shown in [2] that the offensive alliance number of a graph of order $n \geq 2$ is bounded by

$$a_o(\Gamma) \leq \left\lfloor \frac{2n}{3} \right\rfloor \tag{1}$$

and

$$a_o(\Gamma) \leq \left\lfloor \frac{\gamma(\Gamma) + n}{2} \right\rfloor, \tag{2}$$

where $\gamma(\Gamma)$ denotes the domination number of Γ, and the strong offensive alliance number of a graph of order $n \geq 3$ is bounded by

$$a_{\hat{o}}(\Gamma) \leq \left\lfloor \frac{5n}{6} \right\rfloor. \tag{3}$$

It is clear that $a_o(\Gamma) \leq \gamma_o(\Gamma)$ and $a_{\hat{o}}(\Gamma) \leq \gamma_{\hat{o}}(\Gamma)$. In this paper we show a new proof technique for the above results and we obtain new bounds for $\gamma_o(\Gamma)$ and $\gamma_{\hat{o}}(\Gamma)$. We suggest that the "power" of an offensive alliance is greater if the subgraph induced by the alliance is connected than if it is not. Following this idea, the last section of this paper is devoted to the study of *connected alliances*, i.e., alliances whose induced subgraphs are connected.

2. Bounding above the global offensive alliance number

The following theorem shows upper bounds for the global alliance number. We emphasize that bounds (ii) and (iii) of Theorem 1 were already proven in [2]. Here we present a new techniques of proof for these results.

Theorem 1. *For all connected graphs Γ of order $n \geq 2$,*

(i) $\gamma_o(\Gamma) \leq \min\{n - \alpha(\Gamma), \lfloor \frac{n+\alpha(\Gamma)}{2} \rfloor\}$, *where $\alpha(\Gamma)$ denotes the independence number of Γ;*
(ii) *([2] Section 3, Theorem 1)* $\gamma_o(\Gamma) \leq \lfloor \frac{2n}{3} \rfloor$;
(iii) *([2] Section 3, Observation 11)* $\gamma_o(\Gamma) \leq \lfloor \frac{\gamma(\Gamma)+n}{2} \rfloor$, *where $\gamma(\Gamma)$ denotes the domination number of Γ;*
(iv) $\gamma_o(\Gamma) \leq \lfloor \frac{n(2\mu-\delta)}{2\mu} \rfloor$, *where μ denotes the Laplacian spectral radius[1] of Γ and δ denotes its minimum degree.*

[1] The Laplacian spectral radius of Γ is the largest Laplacian eigenvalue of Γ.