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Note

A note on list improper coloring of plane graphs[☆]

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ABSTRACT

A list-assignment L to the vertices of G is an assignment of a set $L(v)$ of colors to vertex v for every $v \in V(G)$. An $(L, d)^*$-coloring is a mapping ϕ that assigns a color $\phi(v) \in L(v)$ to each vertex $v \in V(G)$ such that at most d neighbors of v receive color $\phi(v)$. A graph is called $(k, d)^*$-choosable, if G admits an $(L, d)^*$-coloring for every list assignment L with $|L(v)| \geq k$ for all $v \in V(G)$. In this note, it is proved that every plane graph, which contains no 4-cycles and l-cycles for some $l \in \{8, 9\}$, is $(3, 1)^*$-choosable.

1. Introduction

Graphs considered in this note are finite, simple and undirected. Unless stated otherwise, we follow the notation and terminology in [1].

For a plane graph G, we denote its vertex set, edge set, face set, and minimum degree by $V(G)$, $E(G)$, $F(G)$ and $\delta(v)$, respectively. For a vertex v, $d_G(v)$ and $N_G(v)$ denote its degree and the set of its neighbors in G, respectively.

A k-vertex (or k-face) is a vertex (or a face) of degree k, a k^--vertex (or k^--face) is a vertex (or a face) of degree at most k, and a k^+-vertex (or k^+-face) is defined similarly.

Two faces of a plane graph are said to be adjacent if they have at least one common boundary edge. For $x \in V(G) \cup F(G)$, we use $F_k(x)$ and $V_k(x)$ to denote the set of all k-faces and k-vertices that are incident or adjacent to x, respectively. For $f \in F(G)$, we write $f = [u_1u_2 \cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are on the boundary of f in a clockwise order. A k-face is called an $(m_1, m_2, \ldots, m_k)$-face if $d(u_i) = m_i$ for $i = 1, 2, \ldots, k$.

A k-coloring of G is a mapping ϕ from $V(G)$ to a set of size k such that $\phi(x) \neq \phi(y)$ for any adjacent vertices x and y. A graph is k-colorable if it has a k-coloring.

A list-assignment L to the vertices of G is an assignment of a set $L(v)$ of colors to vertex v for every $v \in V(G)$. If G has a coloring ϕ such that $\phi(v) \in L(v)$ for all vertices v, then we say that G is L-colorable or ϕ is an L-coloring of G. We say that G is k-list colorable (or k-choosable) if it is L-colorable for every list-assignment L satisfying $|L(v)| = k$ for all vertices v.

A graph G is k-colorable with deficiency d, or simply $(k, d)^*$-colorable, if the vertices of G can be colored with k colors so that each vertex has at most d neighbors receiving the same color as itself. A $(k, 0)^*$-coloring is an ordinary k-coloring. Given a list assignment L, an L-coloring with deficiency d, or an $(L, d)^*$-coloring of G, is a mapping $\phi : V(G) \to \cup_{v \in V(G)} L(v)$ such that $\phi(v) \in L(v)$ and every vertex has at most d neighbors receiving the same color as itself. A graph G is called $(k, d)^*$-choosable, if there exists an $(L, d)^*$-coloring for every list assignment L with $|L(v)| = k$ for all $v \in V(G)$.

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The concept of list improper coloring was first independently introduced by Škrekovski [5], and Eaton and Hull [2]. They proved that every plane graph is $(3,2)^*$-choosable and every outerplanar graph is $(2,2)^*$-choosable. Let $g(G)$ denote the girth of a graph G, i.e., the length of a shortest cycle. Škrekovski [7] proved that every planar graph G is $(2,1)^*$-choosable if $g(G) \geq 9$, $(2,2)^*$-choosable if $g(G) \geq 7$, $(2,3)^*$-choosable if $g(G) \geq 6$, and $(2,d)^*$-choosable if $g(G) \geq 5$ and $d \geq 4$. In [6], Škrekovski proved that every plane graph without 3-cycles is $(3,1)^*$-choosable. In [4] it was proved that every plane graph without 4-cycles and l-cycles for some $l \in \{5,6,7\}$ is $(3,1)^*$-choosable. In [10], it is proved that every toroidal graph without adjacent triangles is $(4,1)^*$-choosable. In [9], it is proved that every plane graph with neither adjacent triangles nor 5-cycles is $(3,1)^*$-colorable. Interested readers may read [3,8] for more results and references.

In this note, we will show that every plane graph, which contains no 4-cycles and l-cycles for some $l \in \{8,9\}$, is $(3,1)^*$-choosable. This can be regarded as a complement to the result of [4].

2. Main results and proofs

In order to prove our theorems, we first present two useful lemmas:

Lemma 2.1 ([4]). *Let G be a graph and $d \geq 1$ an integer. If G is not $(k,d)^*$-choosable but every subgraph of G with fewer vertices is, then the following facts hold:*

(1) $\delta(G) \geq k$.
(2) *If $u \in V(G)$ is a k-vertex and v is a neighbor of u, then $\delta(v) \geq k+d$.*

Lemma 2.2 ([4]). *Let G be a graph such that G is not $(k,d)^*$-choosable but every subgraph of G with fewer vertices is. If $\delta(u) \leq k+d$ for a given $u \in V(G)$, then $\delta(v) \geq k+d$ for some $v \in N(u)$.*

Theorem 2.1. *Let G be a plane graph without any cycles of length in $\{4,8\}$, then G is $(3,1)^*$-choosable.*

Proof. Suppose that G is a counterexample with the fewest vertices. First assume that G is 2-connected, thus the boundary of each face of G forms a cycle, and each vertex of G is incident to exactly $d(v)$ distinct faces. Every subgraph H of G with fewer vertices is still plane and without 4, 8-cycles, so H is $(3,1)^*$-choosable. Let L be an arbitrary list of G satisfying $|L(v)| = 3$ for all $v \in V(G)$. The following facts hold for G.

(a) $\delta(G) \geq 3$.
(b) G does not contain two adjacent 3-vertices.
(c) G does not contain a 4-face or two adjacent 3-faces.
(d) G does not contain a (3,4,4)-face.

Fact (b) implies that $|V_3(f)| \leq \lfloor d(f)/2 \rfloor$ for all $f \in F(G)$. Fact (c) implies that $|F_3(v)| \leq \lfloor d(v)/2 \rfloor$ for all $v \in V(G)$. The proof of Fact (d) goes as follows. Suppose to the contrary that G contains a $(3,4,4)$-face $[uvw]$ such that $d(u) = 3$ and $d(v) = d(w) = 4$. By the minimality of G, $G - \{u,v,w\}$ has an $(L,1)^*$-list coloring ϕ. Define $L'(x) = L(x) - A(x)$ for every $x \in \{u,v,w\}$, where $A(x)$ denotes the set of colors that ϕ assigns to the neighbors of x in $G - \{u,v,w\}$. Thus $|L'(u)| \geq 2$, $|L'(v)| \geq 1$, $|L'(w)| \geq 1$. An $(L',1)^*$-coloring of the 3-cycle $uvwu$ can easily be constructed. Hence, G is $(L,1)^*$-colorable, this contradicts the choice of G.

We denote a weight function w on $V(G) \cup F(G)$ by letting $w(v) = d(v) - 4$ for $v \in V(G)$ and $w(f) = d(f) - 4$ for $f \in F(G)$. Applying Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ and the Handshaking Lemmas for vertices and faces of a plane graph, we have

$$\sum_{x \in V \cup F} w(x) = -8.$$

If we obtain a new weight $w^*(x)$ for all $x \in V \cup F$ by transferring weights from one element to another, then we also have $\sum w^*(x) = -8$. If these transfers result in $w^*(x) \geq 0$ for all $x \in V \cup F$, then we get a contradiction and the Theorem is proved.

Now we list our discharging rules.

(R1) For every vertex v with $d(v) \geq 5$, we transfer $\frac{1}{3}$ from v to each adjacent 3-face.
(R2) For every 3-face f, we transfer $\frac{1}{3}$ from f to each incident 3-vertex.
(R3) For every face f with $d(f) \geq 5$, we transfer $\frac{1}{3}$ from f to each incident 3-vertex, $\frac{1}{3}$ from f to each adjacent 3-face.
(R4) For every face f with $d(f) \geq 6$, we transfer $\frac{1}{9}$ from f to each adjacent 5-face.

It remains to show that the resulting weight w^* satisfies $w^*(x) \geq 0$ for all $x \in V \cup F$.

It is evident that $w^*(x) = w(x) = 0$ for all $x \in V \cup F$ with $d(x) = 4$. Let $v \in V(G)$. By (a), $d(v) \geq 3$. If $d(v) = 3$, then, by R2 and R3, $w^*(v) \geq w(v) + 3 \times \frac{1}{3} = 0$. If $d(v) \geq 5$, then by (c) and R1, $w^*(v) = w(v) - \frac{1}{3} \times |F_3(v)| \geq w(v) - \frac{1}{3} \times \lfloor \frac{d(v)}{2} \rfloor \geq 0$. Now let $f \in F(G)$. First assume $d(f) = 3$, note that f is adjacent to 5^+-faces. If f is incident to just one 3-vertex, then by (d), the boundary of f contains 5^+-vertex, thus by R1 to R3, $w^*(f) \geq w(f) + 4 \times \frac{1}{3} - \frac{1}{3} = 0$. If f is not incident to any 3-vertex,