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Stability of two player game structures

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Abstract

We have extended a two player game-theoretical model proposed by V. Gurvich [To theory of multi-step games, USSR Comput. Math and Math. Phys. 13 (1973)] and H. Moulin [The Strategy of Social Choice, North Holland, Amsterdam, 1983]: All the considered *game situations* are framed by the same *game structure*. The structure determines the families of *potential decisions* of the two players, as well as the subsets of *possible outcomes* allowed by pairs of such choices. To be a *solution of a game*, a pair of decisions has to determine a *(pure) functional equilibrium* of the *situational* pair of *payoff mappings* which transforms the *realized outcome* into *real-valued rewards* of the players. Accordingly we understand that a structure is *stable*, if it admits functional equilibria for all possible game situations; and that it is *complete*, if every situation that only *partitions the potential outcomes*, is *dominated* by one of the players. We have generalized and strengthened a theorem by V. Gurvich [Equilibrium in pure strategies, Soviet Math. Dokl. 38 (1989)], proving that a *proper* structure is *stable* iff it is *complete*. Additional results provide game-theoretical insight that focuses the inquiry on the *complexity* of the *stability decision problem*; in particular, for *coherent* structures.

These results also have combinatorial importance because every structure is characterized by a pair of *hypergraphs* [C. Berge, Graphes et Hypergraphes, Dunod, 1970] over a common *ground set*. The structure is *dual* (*complete*/*coherent*) iff the *clutter* of one hypergraph equals (includes/is included in) the *blocker* of the other one. So, for non-void coherent structures, the *stability decision problem* is equivalent to the much studied subexponential [M.L. Fredman, L. Khachiyan, On the complexity of dualization of monotone disjunctive normal forms, J. Algorithms 21 (1996)] *hypergraph duality decision problem*.

Keywords: Game form; Equilibrium; Stable structure; Hypergraph transversal; Duality problem; Polynomial reduction

1. Introduction

Since its beginning [3] game theory has had difficulties in defining what should be considered a *solution of a game*. Because even for the most appealing formalizations, there are *games* that have *no solution*; see, for instance, [12]. Such *unsolvable games* would typically frustrate involved players who could not gasp the tempting rewards. Therefore they also worried game theoreticians. This is probably the reason why some authors began to qualify the original game-theoretical convictions, and started to visualize *structures* that coin or delimit the *universe of situational games* that may take place, as well as what should be considered their *solutions*. Then it became natural to *blame* the prevailing

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structures for the *frustrations* they were not able to prevent. It became sensible to ask, what *particular properties* such a *structure* must have, so that *all possible games* permit solutions. According to some authors [13,9] only such a *structure* may considered to be *stable*; otherwise, *unsettled game situations*, through the *frustrations* they produce, would react against their *structural determinants* and *destabilize* them.

For this paper's purpose we adopt a *categorical formal version* of this point of view: We assume that a structure essentially is a *solving principle* put into practice; and that it therefore gets questioned by any *potential failure of the principle*. *Universal principles* are like *mathematical conjectures*: they get *disproved by counterexamples*. So any potential situation that resists solution, questions the structure; even if the situation does not really take place. Only *universal solvability* guarantees *stability* of the structure.

In this paper we consider such a *structural stability problem*. We are going to build on a game-theoretical model that adopts the classical *Nash equilibrium (in pure strategies)* solution concept; and assumes that the games that can take place in various situations, are all framed by a given *game form* [7,8,2,10]. A game form is called *Nash-solvable* if all the games that the form allows have Nash equilibria. A peak result of this inquiry – Theorem 1 of [10] – then proves that if the game form, structurally, only includes two players, then the game form will be Nash-solvable iff the game form is *tight*; i.e., iff *the pair of hypergraphs that the form determines* – one for each player – is *dual*. We extend this game-theoretic model from game forms to *structures* that are *pairs of hypergraphs over a common finite ground set*. So this class of structures includes, as special cases, those *determined by game forms* [10]. Since now all such pairs of hypergraphs are possible structures, this model also allows us to express game-theoretical reformulation of all the questions raised by the *hypergraph duality theory* [6,11,14,4].

The paper is organized as follows: In Section 2 we first gather some *classical results on hypergraphs* which we shall later use to develop our game-theoretical considerations. In Section 3 we introduce to the indispensable *game-theoretical background*. In Section 4 we recycle *classical blocker theory* [5] to analyze the *solvability of structures* that are exposed to *antagonistic games*. In Section 5 we define *structural stability* and derive the new *minimal transversal* considerations that sustain our main results. Finally, in Section 6, we prove that one can restrict the attention to *proper game structures*, or even to game forms.

2. Hypergraphs

In all what follows, A will be a given finite *ground set*.

Definition. A *hypergraph* on the ground set A, is a family $\mathcal{H}$ of subsets of A. $\mathcal{H}$ is called *proper*, if $\mathcal{H} \neq \emptyset$ and $\emptyset \notin \mathcal{H}$. The *domain* of $\mathcal{H}$ is $\bigcup\{X \in \mathcal{H}\} \subseteq A$. If it equals A, then $\mathcal{H}$ has *full domain*. The original definition [1] required of hypergraphs to be proper and have full domain. Although such proper hypergraphs are the ones that will most interest us, like other authors [4] we will not restrict the notion. The *size* of $\mathcal{H}$ is $\kappa(\mathcal{H}) := |A| \cdot |\mathcal{H}| \in \mathbb{N}$.

Definition. Given a hypergraph $\mathcal{H}$, let $\nu(\mathcal{H}) := \{W \subseteq A; \exists X \in \mathcal{H}, X \subseteq W\}$ denote the family of subsets of A that are *responded* [14] by (members of) $\mathcal{H}$ — or, according to other authors, the *clutter* of $\mathcal{H}$. Note that $\nu \circ \nu = \nu$.

Definition. A *(hypergraph) structure on A* is a cartesian product $\mathcal{G} := \mathcal{H} \times \mathcal{K}$ of two hypergraphs over the same *ground set A*. $\mathcal{G}$ is called *proper*, if $\mathcal{H}$ and $\mathcal{K}$ are proper. $\mathcal{G}$ has *unique domain* [6], if $\bigcup\{X \in \mathcal{H}\} = \bigcup\{Y \in \mathcal{K}\}$. The *size* of the structure $\mathcal{G}$ is $\kappa(\mathcal{G}) := |A| \cdot (|\mathcal{H}| + |\mathcal{K}|) \in \mathbb{N}$.
The structure $\mathcal{G}$ is called *coherent* [14], if for all partitions (W, Z) of A, either $W \notin \nu(\mathcal{H})$ or $Z \notin \nu(\mathcal{K})$.
This property can be decided in *polynomial time* – i.e., with a computational effort that can be bounded by a polynomial in $\kappa(\mathcal{G})$ – since, as is easy to see, it holds iff $\forall(X, Y) \in \mathcal{H} \times \mathcal{K}, X \cap Y \neq \emptyset$.
The structure $\mathcal{G}$ is called *complete* [14], if for all partitions (W, Z) of A, either $W \in \nu(\mathcal{H})$ or $Z \in \nu(\mathcal{K})$. The corresponding *completeness decision problem* is *coNP-complete* [14].
The structure $\mathcal{G}$ is called *dual*, if it is coherent and complete. The corresponding *duality decision problem* can be solved in *subexponential time* [6].
Note that each of the three considered *structural properties* – like all that will interest us in what follows – is *symmetric*: it holds for $\mathcal{H} \times \mathcal{K}$ iff it holds for $\mathcal{K} \times \mathcal{H}$.

Definition. Given a hypergraph $\mathcal{H}$, let $\tau(\mathcal{H}) := \{Z \subseteq A; \forall X \in \mathcal{H}, X \cap Z \neq \emptyset\}$ denote the family of all *transversal* of $\mathcal{H}$ – or, according to other authors, the *blocker* of $\mathcal{H}$.