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On the connectivity of close to regular multipartite tournaments

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Abstract

If x is a vertex of a digraph D, then we denote by $d^+(x)$ and $d^-(x)$ the outdegree and the indegree of x, respectively. The global irregularity of a digraph D is defined by $i_g(D)=\max\{d^+(x), d^-(x)\}-\min\{d^+(y), d^-(y)\}$ over all vertices x and y of D (including $x=y$) and the local irregularity of a digraph D is $i_l(D)=\max|d^+(x)-d^-(x)|$ over all vertices x of D. Clearly, $i_l(D)\leqslant i_g(D)$. If $i_g(D)=0$, then D is regular and if $i_g(D)\leqslant 1$, then D is almost regular.

A c-partite tournament is an orientation of a complete c-partite graph. Let $V_1, V_2, \ldots, V_c$ be the partite sets of a c-partite tournament such that $|V_1|\leqslant|V_2|\leqslant\cdots\leqslant|V_c|$. In 1998, Yeo proved

$$\kappa(D)\geqslant\left\lceil\frac{|V(D)|-|V_c|-2i_l(D)}{3}\right\rceil$$

for each c-partite tournament D, where $\kappa(D)$ is the connectivity of D. Using Yeo's proof, we will present the structure of those multipartite tournaments, which fulfill the last inequality with equality. These investigations yield the better bound

$$\kappa(D)\geqslant\left\lceil\frac{|V(D)|-|V_c|-2i_l(D)+1}{3}\right\rceil$$

in the case that $|V_c|$ is odd. Especially, we obtain a 1980 result by Thomassen for tournaments of arbitrary (global) irregularity. Furthermore, we will give a shorter proof of the recent result of Volkmann that

$$\kappa(D)\geqslant\left\lceil\frac{|V(D)|-|V_c|+1}{3}\right\rceil$$

for all regular multipartite tournaments with exception of a well-determined family of regular $(3q+1)$-partite tournaments. Finally we will characterize all almost regular tournaments with this property.

Keywords: Multipartite tournaments; Almost regular multipartite tournaments; Connectivity

1. Terminology and introduction

In this paper all digraphs are finite without loops and multiple arcs. The vertex set and arc set of a digraph D is denoted by $V(D)$ and $E(D)$, respectively. If xy is an arc of a digraph D, then we write $x\to y$ and say that x *dominates* y, and if

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X and Y are two disjoint vertex sets or subdigraphs of D such that every vertex of X dominates every vertex of Y, then we say that X *dominates* Y, denoted by $X \rightarrow Y$. Furthermore, $X \rightsquigarrow Y$ denotes the fact that there is no arc leading from Y to X. For the number of arcs from X to Y we write $d(X, Y)$. Furthermore, let $E(X, Y) = d(X, Y) + d(Y, X)$. If D is a digraph, then the *out-neighborhood* $N_D^+(x) = N^+(x)$ of a vertex x is the set of vertices dominated by x and the *in-neighborhood* $N_D^-(x) = N^-(x)$ is the set of vertices dominating x. Therefore, if there is the arc $xy \in E(D)$, then y is an *outer neighbor* of x and x is an *inner neighbor* of y. The numbers $d_D^+(x) = d^+(x) = |N^+(x)|$ and $d_D^-(x) = d^-(x) = |N^-(x)|$ are called the *outdegree* and *indegree* of x, respectively. For a vertex set X of D, we define $D[X]$ as the subdigraph induced by X. If we replace in a digraph D every arc xy by yx, then we call the resulting digraph the *converse* of D, denoted by D^{-1}.

There are several measures of how much a digraph differs from being regular. In [11], Yeo defines the *global irregularity* of a digraph D by

$$i_g(D) = \max_{x \in V(D)} \{d^+(x), d^-(x)\} - \min_{y \in V(D)} \{d^+(y), d^-(y)\}$$

and the *local irregularity* by $i_l(D) = \max\{|d^+(x) - d^-(x)| \,|\, x \in V(D)\}$. Clearly $i_l(D) \leqslant i_g(D)$. If $i_g(D) = 0$, then D is *regular* and if $i_g(D) \leqslant 1$, then D is called *almost regular*.

A digraph D is *strongly connected* or *strong* if, for each pair of vertices u and v, there are a directed path from u to v, and a directed path from v to u in D. A digraph D with at least $k+1$ vertices is *k-connected* if for any set A of at most $k-1$ vertices, the subdigraph $D - A$ obtained by deleting A is strong. The *connectivity* of D, denoted by $\kappa(D)$, is then defined to be the largest value of k such that D is k-connected. If S is a set of vertices of D such that the subdigraph $D - S$ is not strongly connected, then S is called a *separating set*.

A *c-partite* or *multipartite tournament* is an orientation of a complete c-partite graph. A *tournament* is a c-partite tournament with exactly c vertices. A *semicomplete multipartite digraph* is obtained by replacing each edge of a complete multipartite graph by an arc or by a pair of two mutually opposite arcs. If $V_1, V_2, \ldots, V_c$ are the partite sets of a c-partite tournament D and the vertex x of D belongs to the partite set V_i, then we define $V(x) = V_i$. If D is a c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| \leqslant |V_2| \leqslant \cdots \leqslant |V_c|$, then $|V_c| = \alpha(D)$ is the independence number of D, and we define $\gamma(D) = |V_1|$. Note that especially for tournaments, the global and the local irregularity have the same value. Hence, in this case we shortly speak of the *irregularity* $i(T)$ of a tournament T.

In 1998, Yeo [10] proved the following useful bound.

Theorem 1.1 (*Yeo [10]*). *Let D be a c-partite tournament. Then*

$$\kappa(D) \geqslant \left\lceil \frac{|V(D)| - \alpha(D) - 2i_l(D)}{3} \right\rceil. \tag{1}$$

In general, this bound cannot be improved as the following example demonstrates (see also [6]).

Example 1.2 (*Volkmann [6]*). Let $q \geqslant 1$ be an integer, and let $c = 3q + 1$. We define the families $\mathscr{F}_q$ of c-partite tournaments with the partite sets $W_1, W_2, \ldots, W_q$ and

$$W_{q+1} = A_{q+1} \cup B_{q+1},\ W_{q+2} = A_{q+2} \cup B_{q+2}, \ldots, W_c = A_c \cup B_c$$

with $2|A_i| = 2|B_i| = |W_j| = 2t$ for $i = q+1, q+2, \ldots, c$ and $j = 1, 2, \ldots, q$ as follows. The partite sets $W_1, W_2, \ldots, W_q$ induce a $t(q-1)$-regular q-partite tournament H, the sets $A_{q+1}, A_{q+2}, \ldots, A_c$ induce a tq-regular $(2q+1)$-partite tournament A, and the sets $B_{q+1}, B_{q+2}, \ldots, B_c$ induce a tq-regular $(2q+1)$-partite tournament B. In addition, let $H \rightarrow A \rightsquigarrow B \rightarrow H$. Obviously, if $D \in \mathscr{F}_q$, then D is a $3qt$-regular c-partite tournament with the separating set $V(H)$ and thus $\kappa(D) = 2qt = q\alpha(D)$.

Since Yeo's result is often used to solve problems depending on the global irregularity, it would be interesting to solve the following general problem.

Problem 1.3. For each integer $i \geqslant 0$ find all multipartite tournaments D with $i_g(D) = i$ and the property that

$$\kappa(D) = \left\lceil \frac{|V(D)| - |V_c| - 2i}{3} \right\rceil.$$