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Note

On the variance of Shannon products of graphs

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Abstract

We study the combinatorial problem of finding an arrangement of distinct integers into the d-dimensional N-cube so that the maximal variance of the numbers on each ℓ-dimensional section is minimized. Our main tool is an inequality on the Laplacian of a Shannon product of graphs, which might be a subject of independent interest. We describe applications of the inequality to multiple description scalar quantizers (MDSQ), to get bounds on the bandwidth of products of graphs, and to balance edge-colorings of regular, d-uniform, d-partite hypergraphs.

Keywords: Labeling; Variance; Graph product

1. Introduction

Applications of multiple description scalar quantizers (MDSQ) arise in speech and video coding over packet-switched networks, where packet losses can result in a degradation in signal quality, see [8,10]. Here it is desired to send messages across multiple independent channels in such a way that certain guarantees on the reconstructed message fidelity apply if one or more channels are broken, and it is also desired not to decrease the total rate of communication too greatly, see [8,10].

The model translates to the combinatorial problem of finding an arrangement of the integers into $\mathbb{Z}^d$ so that each line of $\mathbb{Z}^d$ contains exactly N numbers, such that the variance of the numbers in each ℓ-dimensional section is minimized, where a *line* corresponding to fixed integers $a_1, \dots, a_d$ and direction k is the set $\{\langle a_1, \dots, a_{k-1}, i, a_{k+1}, \dots, a_d \rangle \in \mathbb{Z}^d \ : \ i \in \mathbb{Z}\}$, and an *$\ell$-dimensional section* defined similarly, here all but $d - \ell$ coordinates are fixed. Then minimizing the distortion given the rate amounts to finding, for a given N, d and ℓ, an arrangement with the smallest possible *variance* $Var_\infty(N, d, \ell)$, which is defined formally as $Var_\infty(N, d, \ell) = \max\{(1/N) \sum (\bar{X} - X_i)^2\}$, where the maximum is over all ℓ-dimensional sections of $\mathbb{Z}^d$, and the summation is over all elements X_i of a section, and $\bar{X}$ is the mean of these numbers (in one section). This problem was considered in [10,2]. In [2] it was proved that $(1/60)N^4 \leqslant Var_\infty(N, 2, 1) \leqslant (1/54)N^4 + O(N^3)$.

One of the key observations in [2] was (in the case $d = 2$ and $\ell = 1$) that to obtain good bounds for $Var_\infty(N, d, \ell)$, the following problem needed to be considered: Write the integers $1, \dots, N^d$ into an N^d cuboid such that the maximal

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variance of the numbers appearing in an ℓ-dimensional section of the cuboid is as small as possible. We denote this minimum by $Var(N, d, \ell)$.

For this problem the bounds $(1 + 10^{-5})N^4/24 < Var(N, 2, 1) < (1/22)N^4$ were obtained in [2]. To achieve the $(1/24)N^4$ lower bound, an inequality was used, whose extension is the main aim of this note. In Section 2, as a warm-up, we prove it for two-dimensional rectangles, in Section 3 we shall state and prove it for any dimension. We also prove a generalization to the Laplacian of a Shannon product of graphs. In Section 4 we state some conclusions for bandwidth of products of cliques, and in Section 5 we give bounds on a problem concerning edge-colorings of hypergraphs. Note that isoperimetric problems on the products of regular graphs were studied in the literature, see for example [3].

2. A lower bound for the variance in a rectangle

For the sake of completeness we recall a lemma from [2], considering N by M matrices: As usual, let the *variance* of a list $(X_1, X_2, \ldots, X_n)$ of real numbers be denoted by

$$Var(X_1, X_2, \ldots, X_n) = \frac{1}{n}\sum_i (X_i - \overline{X})^2,$$

where $\overline{X} = \frac{1}{n}\sum_i X_i$ is the *mean* of $(X_1, X_2, \ldots, X_n)$. The following identity motivates many of our forthcoming definitions:

$$Var(X_1, X_2, \ldots, X_n) = \frac{1}{n^2}\sum_{i<j} (X_i - X_j)^2. \tag{1}$$

Theorem 2.1. *Let $X_{i,j}$ $(1 \leqslant i \leqslant N, 1 \leqslant j \leqslant M)$ denote the elements of an N-by-M matrix. Then*

$$Var(X_{1,1}, X_{1,2}, \ldots, X_{N,M}) \leqslant \frac{1}{N}\sum_i Var(X_{i,1}, X_{i,2}, \ldots, X_{i,M}) + \frac{1}{M}\sum_j Var(X_{1,j}, X_{2,j}, \ldots, X_{N,j}). \tag{2}$$

Proof. Substitute the definitions of all the variances into (2), multiply by N^2M^2, and move all terms to the right side. It is easy to check that the coefficients of the monomials on the right-hand side are as follows. The coefficients of the terms of form $X_{i,j}^2$ (where i and j need not be distinct) are $(N-1)(M-1)$, the coefficients of the terms of form $X_{i,j}X_{\ell,k}$ (with $i \neq \ell$ and $j \neq k$) are 2, and the coefficients of the remaining terms of form $X_{i,j}X_{i,k}$ are $2(1-N)$ and of $X_{i,j}X_{\ell,j}$ are $2(1-M)$. There are no other terms.

This transformation shows that our assertion is equivalent to stating that a certain quadratic form in the variables $X_{i,j}$ is positive semidefinite. Let G denote the NM-by-NM matrix whose rows (resp. columns) are labeled with the variables $X_{i,j}$ that represents the quadratic form in question. The entries of G can be read off from the calculation above. The diagonal entries are all $(N-1)(M-1)$. The off-diagonal entry corresponding to row $X_{i,j}$ and column $X_{\ell,k}$ is the half of the coefficients described above.

The matrix G is the Kronecker (tensor) product of an N-by-N matrix K_N and an M-by-M matrix K_M whose diagonal elements are equal to $N-1$, (resp. $M-1$) and whose other elements are -1. Since in general the matrix K_N has eigenvalues 0 (with multiplicity 1) and N (with multiplicity $N-1$ times), the matrix G has eigenvalues 0 ($N+M-1$ of them) and NM ($(N-1)(M-1)$ of them). This proves our result. □

Remark. An alternative proof is the following. Denote $A_{i,j} := X_{i,j} - (1/N)\sum_{i=1}^{N} X_{i,j}$. Let X be a randomly (uniformly) chosen element of the matrix, Y be a randomly (uniformly) chosen row, and Z be a column (to clarify notation; the first row consists of $X_{1,1}, X_{1,2}, \ldots, X_{1,M}$). With this notation the relation (2) is equivalent to the following:

$$Var(X) \leqslant \mathbb{E}[Var(X|Y)] + \mathbb{E}[Var(X|Z)]. \tag{3}$$