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On powers of graphs of bounded NLC-width (clique-width)☆

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Abstract

Given a graph G, the graph G^l has the same vertex set and two vertices are adjacent in G^l if and only if they are at distance at most l in G. The l-coloring problem consists in finding an optimal vertex coloring of the graph G^l, where G is the input graph. We show that, for any fixed value of l, the l-coloring problem is polynomial when restricted to graphs of bounded NLC-width (or clique-width), if an expression of the graph is also part of the input. We also prove that the NLC-width of G^l is at most $2(l+1)^{\mathrm{nlcw}(G)}$.

Keywords: Clique-width; NLC-width; Coloring; Power graph; Polynomial

1. Introduction

Many generalizations of the coloring problem of graphs were proposed as models for real-life applications. One of them is the radio-coloring problem (also called λ-coloring or $L(2,1)$-coloring), in which the vertices of the graph represent transmitters, and two vertices are adjacent if and only if the two transmitters are very close. Each vertex must receive an integer frequency, such that adjacent vertices (very close transmitters) get frequencies differing by at least two units, and vertices at distance two in the graph (close transmitters) get different frequencies. The aim is to minimize the range of the used frequencies, i.e., we search for the minimum λ such that G admits a radio-coloring with frequencies between 0 and λ. Let us call $\lambda_{2,1}$ this minimum. In general case, $L(d_1,d_2)$-coloring means that for adjacent vertices the colors must differ by at least d_1 and for vertices in distance 2 they must differ by at least d_2. Another generalization of the classical coloring is the l-coloring problem, i.e. the coloring of G^l, where G is the input graph. Here $G^l=(V,E^l)$ denotes the lth power of G: two vertices are adjacent in G^l if and only if they are at distance at most l in G. The radio-coloring and the l-coloring problems are related. Every radio-coloring is also a 2-coloring. On the other hand, by multiplying the values of colors in a 2-coloring by two, we easily obtain a radio-coloring. Thus $\lambda_{1,1}\leqslant\lambda_{2,1}\leqslant 2\lambda_{1,1}$, where $\lambda_{1,1}$ denotes the minimum number of colors necessary for a 2-coloring (or, equivalently, for a $L(1,1)$-coloring). Hence an algorithm solving the 2-coloring problem for a class of graphs also provides a 2-approximation for the radio-coloring problem. Both problems are NP-hard for general graphs [1,6]. The l-coloring problem is polynomial for graphs of tree-width at most k, for any fixed k and l [12].

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In this paper we study the NLC-width of G^l and the l-coloring problem on graphs of bounded NLC-width (or, equivalently, of bounded clique-width).

The class of k-node label controlled graphs (denoted NLC_k) is recursively defined using k vertex labels and three operators: $\bullet_i$, $\circ_R$, $\times_S$. The operation $\bullet_i$ creates a graph with a single vertex labelled $i \in [k]$, where $[k]=\{1, 2, \ldots, k\}$. The operation $\circ_R$, with R being a total function $[k] \to [k]$, replaces every instance of label i with $R(i)$. The binary operator $\times_S$, with $S \subseteq [k] \times [k]$, creates the disjoint union of given graphs G, H and adds edges between the vertices of G labelled i and the vertices of H labelled j, for every $(i, j) \in S$. The NLC-width of an unlabelled graph G is the minimum value of k such that G, with some appropriate labelling, is in NLC_k. The clique-width parameter is defined in a very similar way. The class of graphs CWD_k is recursively defined using four operators. The operator $\bullet_i$, like above, creates a graph with a unique vertex labelled i. The unary operator $\rho_{i\to j}$ relabels all instances of label i with label j and the unary operator $\eta_{i,j}$ adds all the edges xy with x (resp. y) of label i (resp. j). The binary operator $\oplus$ constructs the disjoint union of two graphs. The clique-width and the NLC-width of a graph differ by at most a factor of two. All the results of this paper could be stated either in terms of clique-width or using NLC-width. Nevertheless, we consider that in our case it was more convenient to use NLC-width, in order to simplify some notations.

Many NP-hard problems become polynomially solvable for graphs of bounded NLC-width (clique-width), if an expression of the graph, using the operators above, is part of the input. The classical coloring problem is one of them [5,8]. Let us note that graphs of bounded tree-width are a particular case of graphs of bounded clique-width, more precisely if G has tree-width at most k then $\mathrm{cwd}(G) \leqslant 2^{k+1}+1$ [4]. Several other classes of graphs (distance–hereditary graphs, P_4-sparse and P_4-tidy graphs, …) are known to have bounded clique-width.

Up to now, one of the limits for these decomposition techniques was due to the fact that there were no good algorithms for computing or approximating the clique-width or the NLC-width of a graph. Except for very small values of the parameters (cwd $\leqslant 3$ or nlcw $\leqslant 2$), there was only an $\mathcal{O}(\lg n)$ approximation algorithm which is not sufficient for most of the problems. Recently, Oum and Seymour [9] announced a 3-approximation algorithm for the rank-width of a graph, and this parameter is strongly related to clique-width: rank-width $\leqslant$ cwd $\leqslant 2^{\text{rank-width}+1}$ [9]. The new result might strongly increase the interest for graphs of bounded clique-width.

The initial motivation for this work was the radio-coloring problem for graphs of bounded clique-width, but it turns out that the problem is NP-complete even for graphs of clique-width at most 3 [2]. We show in this article that the l-coloring problem is polynomial when restricted to graphs of bounded NLC-width, if an expression of the graph is part of the input. This result extends the one on graphs of bounded tree-width [12]. As mentioned above, it provides a 2-approximation for the radio-coloring problem on graphs of bounded NLC-width.

After introducing the basic notions in Section 2, we show in Section 4 that, for any graph G of NLC-width k, the NLC-width of G^l is at most $2(l+1)^k$. We think that this result is interesting in its own right, since it shows that, when considering a class of graphs of bounded NLC-width, the NLC-width stays bounded when taking powers. Note that a similar result does not hold for tree-width.

Using the coloring algorithm for graphs of bounded clique-width proposed in [8], this result directly implies that, for fixed k and l, the l-coloring problem is solvable in $\mathcal{O}(n^{2^{4\cdot 2^k \lg(l+1)}+1})$ time when restricted to graphs of clique-width at most k if an expression of G is part of the input. In Section 5 we give an algorithm for the l-coloring problem with a $\mathcal{O}(n^{3\cdot 2^k \lg(l+1)} n^4)$ time bound. Although the complexity remains high, it is considerably lower than the previous one. Moreover, this time bound is to be compared with the existing time bounds for classical coloring problem on graphs with clique-width at most k ($\mathcal{O}(n^{2^{2k+1}})$, see [8]) or with the time bound of the l-coloring problem on partial k-trees ($\mathcal{O}(n^{2^{2(k+1)(l+2)+1}})$, see [12]).

2. Basic definitions

Throughout this paper we consider simple, finite, undirected graphs. Given a graph $G=(V, E)$, we denote by n the number of vertices of G. An edge $\{x, y\}$ will be simply denoted by xy. A path $\mu=[x_1, \ldots, x_l]$ is a sequence of vertices such that $x_i x_{i+1} \in E$, $\forall i$, $1 \leqslant i < l$. The length $|\mu|$ of this path is $l-1$.

The distance between two vertices x and y of G, denoted by $\mathrm{dist}_G(x, y)$, is the length of the shortest path from x to y. Also, $\mathrm{dist}_G(W_1, W_2)$ denotes $\min\{\mathrm{dist}_G(x, y) | x \in W_1, y \in W_2\}$ for two sets of vertices $W_1, W_2 \subseteq V$. We define by $G^l=(V, E^l)$ the lth power of G, i.e., $E^l=\{xy \mid \mathrm{dist}_G(x, y) \leqslant l\}$.