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Simple image set of linear mappings in a max–min algebra☆

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Abstract

For a given linear mapping, determined by a square matrix A in a max–min algebra, the set S_A consisting of all vectors with a unique pre-image (in short: the simple image set of A) is considered. It is shown that if the matrix A is generally trapezoidal, then the closure of S_A is a subset of the set of all eigenvectors of A. In the general case, there is a permutation π, such that the closure of S_A is a subset of the set of all eigenvectors permuted by π. The simple image set of the matrix square and the topological aspects of the problem are also described.

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1. Introduction

Problems in many research areas, such as system theory, graph theory, scheduling, knowledge engineering, can be formulated in a compact way using the language of extremal algebras, in which the addition and multiplication of vectors and matrices is formally replaced by operations of maximum and minimum, or maximum and plus. Systematic approach in this field can be found in [5,6]. Many authors studied the questions of extremal algebras, similar to those of linear algebra, like solvability of the systems of linear equations, linear mappings, independence, regularity, eigenvectors and eigenvalues.

In [10], the following question was posed: given a fuzzy relation R between medical symptoms expressing the action of a drug on patients in a given therapy, what is the greatest invariants of the system? The question leads to the problem of finding the greatest eigenvector of the max–min matrix A corresponding to the fuzzy relation R.

The aim of this paper is to describe the set consisting of all vectors with a unique pre-image (in short: the simple image set) of a given max–min linear mapping. In the above interpretation, a vector with a unique pre-image corresponds to such a patient's state, which results from a unique state preceding the treatment by a given drug. We present a close

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connection of the simple image set with the eigenspace of the corresponding matrix (the set of all fixed points of the mapping). The simple image set of the matrix square and the topological aspects of the problem are described in the last two sections. The questions considered in this paper are analogous to those in [1], where matrices and linear mappings in a max-plus algebra are studied.

2. Notions and notation

By a max–min algebra we mean a linearly ordered set $(\mathscr{B}, \leqslant)$ with the binary operations of maximum and minimum, denoted by $\oplus$ and $\otimes$. For any natural $n > 0$, $\mathscr{B}(n)$ denotes the set of all n-dimensional column vectors over $\mathscr{B}$, and $\mathscr{B}(m, n)$ denotes the set of all matrices of type $m \times n$ over $\mathscr{B}$. We shall use the notation $M = \{1, 2, \ldots, m\}$, $N = \{1, 2, \ldots, n\}$. For $x, y \in \mathscr{B}(n)$, we write $x \leqslant y$, if $x_i \leqslant y_i$ holds for all $i \in N$, and we write $x < y$, if $x \leqslant y$ and $x \neq y$. We say that a vector $x \in \mathscr{B}(n)$ is increasing, if $x_i \leqslant x_j$ holds for every $i, j \in N, i \leqslant j$. The matrix operations over $\mathscr{B}$ are defined with respect to $\oplus$, $\otimes$, formally in the same manner as the matrix operations over any field. In general, $\mathscr{B}$ need not be bounded. We shall denote by $\mathscr{B}^\star$ the bounded algebra derived from $\mathscr{B}$ by adding the least element, or the greatest element (or both), if necessary. If $\mathscr{B}$ itself is bounded, then $\mathscr{B} = \mathscr{B}^\star$. The least element in $\mathscr{B}^\star$ will be denoted by O, the greatest one by I. To avoid the trivial case, we assume $O < I$. (This notation should not be mixed up with the notation for sets $I_j(A, b)$ introduced below.)

Many authors considered the system of linear equations of the form

$$A \otimes x = b, \tag{1}$$

where the matrix $A \in \mathscr{B}(m, n)$ and the vector $b \in \mathscr{B}(m)$ are given, and the vector $x \in \mathscr{B}(n)$ is unknown. It was shown in [4], that the consideration of the solvability of (1) may be reduced to the case when $b_i > O$ for all $i \in M$. In the following we shall use the notation from [7], where the questions of solvability and unique solvability were studied.

The solvability of the equation (1) is closely related to its greatest solution denoted by $\bar{x}(A, b)$. The vector $\bar{x}(A, b) \in \mathscr{B}^\star(n)$ is defined by putting, for every $j \in N$,

$$M_j(A, b) := \{i \in M; a_{ij} > b_i\}, \quad \bar{x}_j(A, b) := \min_{\mathscr{B}^\star}\{b_i; i \in M_j(A, b)\}.$$

The vector $\bar{x} = \bar{x}(A, b)$ is defined correctly, because the minimum in the definition is computed in the upper bounded algebra $\mathscr{B}^\star$. Therefore, every value $\bar{x}_j$ is well-defined, even in the case, when $M_j(A, b)$ is an empty set (then $\bar{x}_j = \min_{\mathscr{B}^\star}\emptyset = I \in \mathscr{B}^\star$).

Lemma 2.1 (*Gavalec [7]*). *Let $x \in \mathscr{B}(n)$ be a solution of the equation $A \otimes x = b$. Then $x \leqslant \bar{x}$ and $\bar{x}$ is a solution of the equation $A \otimes x = b$ in $\mathscr{B}^\star(n)$.*

Theorem 2.2 (*Gavalec [7]*). *Let $A \in \mathscr{B}(m, n)$, $b \in \mathscr{B}(m)$. The equation $A \otimes x = b$ has a solution in $\mathscr{B}(n)$ if and only if $\bar{x}$ is a solution of $A \otimes x = b$ in $\mathscr{B}^\star(n)$.*

The unique solvability of the equation (1) in a general max–min algebra $\mathscr{B}$ is characterized in [7] by a necessary and sufficient condition. The following notation is used: for every $j \in N$,

$$I_j(A, b) := \{i \in M; a_{ij} \geqslant b_i = \bar{x}_j\}, \quad \mathscr{I}(A, b) := \{I_j(A, b); j \in N\},$$

$$K_j(A, b) := \{i \in M; a_{ij} = b_i < \bar{x}_j\}, \quad \mathscr{K}(A, b) := \{K_j(A, b); j \in N\}.$$

If S is a set and $\mathscr{C} \subseteq P(S)$ is a set of subsets of S, we say that $\mathscr{C}$ is a covering of S, if $\bigcup \mathscr{C} = S$, and we say that a covering $\mathscr{C}$ of S is minimal, if $\bigcup(\mathscr{C} - \{C\}) \neq S$ holds for every $C \in \mathscr{C}$.

Theorem 2.3 (*Gavalec [7]*). *Let $A \in \mathscr{B}(m, n)$, $b \in \mathscr{B}(m)$. The equation $A \otimes x = b$ has a unique solution $x \in \mathscr{B}(n)$ if and only if the system $\mathscr{I}(A, b)$ is a minimal covering of the set $M - \bigcup \mathscr{K}(A, b)$.*

The following equivalent formulation of Theorem 2.3 will be used in Section 4.