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Introduction to partially ordered patterns

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Dedicated to the 50th anniversary of my advisor, Einar Steingrímsson

Abstract

We review selected known results on partially ordered patterns (POPs) that include co-unimodal, multi- and shuffle patterns, peaks and valleys ((modified) maxima and minima) in permutations, the Horse permutations and others. We provide several new results on a class of POPs built on an arbitrary flat poset, obtaining, as corollaries, the bivariate generating function for the distribution of peaks (valleys) in permutations, links to Catalan, Narayana, and Pell numbers, as well as generalizations of a few results in the literature including the descent distribution. Moreover, we discuss a q-analogue for a result on non-overlapping segmented POPs. Finally, we suggest several open problems for further research.

Keywords: (Partially ordered) Pattern; Non-overlapping occurrences; Peak; Valley; q-Analogue; Flat poset; Co-unimodal pattern; Bijection; (Exponential bivariate) Generating function; Distribution; Catalan numbers; Narayana numbers; Pell numbers

1. Introduction and background

An occurrence of a *pattern* τ in a permutation π is defined as a subsequence in π (of the same length as τ) whose letters are in the same relative order as those in τ. For example, the permutation 31425 has three occurrences of the pattern 1-2-3, namely the subsequences 345, 145, and 125. *Generalized permutation patterns* (GPs) being introduced in [2] allow the requirement that some adjacent letters in a pattern must also be adjacent in the permutation. We indicate this requirement by removing a dash in the corresponding place. Say, if pattern 2-31 occurs in a permutation π, then the letters in π that correspond to 3 and 1 are adjacent. For example, the permutation 516423 has only one occurrence of the pattern 2-31, namely the subword 564, whereas the pattern 2-3-1 occurs, in addition, in the subwords 562 and 563. Placing "[" on the left (resp., "]" on the right) next to a pattern p means the requirement that p must begin (resp., end) from the leftmost (resp., rightmost) letter. For example, the permutation 32415 contains two occurrences of the pattern [2-13, namely the subwords 324 and 315 and no occurrences of the pattern 3-2-1].

A further generalization of the GPs is *partially ordered patterns* (POPs), where the letters of a pattern form a partially ordered set (poset), and an occurrence of such a pattern in a permutation is a linear extension of the corresponding poset in the order suggested by the pattern (we also pay attention to eventual dashes and brackets). For instance, if we have a poset on three elements labeled by $1'$, 1, and 2, in which the only relation is $1 < 2$ (see Fig. 1), then in an occurrence

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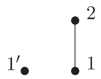

Fig. 1. A poset on three elements with the only relation $1<2$.

of $p=1'\text{-}12$ in a permutation π the letter corresponding to the $1'$ in p can be either larger or smaller than the letters corresponding to 12. Thus, the permutation 31 254 has three occurrences of p, namely 3-12, 3-25, and 1-25.

Let $\mathscr{S}_n(p_1,\ldots,p_k)$ denote the set of n-permutations avoiding simultaneously each of the patterns $p_1,\ldots,p_k$.

The POPs were introduced in [16][2] as an auxiliary tool to study the maximum number of non-overlapping occurrences of *segmented* GPs (SGPs), also known as *consecutive* GPs, that is, the GPs, occurrences of which in permutations form contiguous subwords (there are no dashes). However, the most useful property of POPs known so far is their ability to "encode" certain sets of GPs which provide a convenient notation for those sets and often gives an idea how to treat them. For example, the original proof of the fact that $|\mathscr{S}_n(123,132,213)|=\binom{n}{\lfloor n/2\rfloor}$ took 3 pages [15]; on the other hand, if one notices that $|\mathscr{S}_n(123,132,213)|=|\mathscr{S}_n(11'2)|$, where the letters 1, $1'$, and 2 came from the same poset as above, then the result is easy to see. Indeed, we may use the property that the letters in odd and even positions of a "good" permutation do not affect each other because of the form of $11'2$. Thus, we choose the letters in odd positions in $\binom{n}{\lfloor n/2\rfloor}$ ways, and we must arrange them in decreasing order. We then must arrange the letters in even positions in decreasing order too.

The POPs can be used to encode certain combinatorial objects by restricted permutations. Examples of that are Propositions 10 and 13, as well as several other propositions in [6]. Such encoding is interesting from the point of view of finding bijections, but it also may have applications in enumerating certain statistics. The idea is to encode a set of objects under consideration as a set of permutations satisfying certain restrictions (given by certain POPs); under appropriate encodings, this allows us to transfer the interesting statistics from the original set to the set of permutations, where they are easy to handle. For an illustration of how encodings by POPs can be used, see [19, Theorem 2.4] which deals with POPs in *compositions* rather than in permutations; however but the approach remains the same.

As a matter of fact, some POPs appeared in the literature before they were actually introduced. Thus, the notion of a POP allows us to collect under one roof (to provide a uniform notation for) several combinatorial structures such as *peaks*, *valleys*, *modified maxima* and *modified minima* in permutations, *Horse permutations* and *p-descents* in permutations discussed in Section 2.

This paper is organized as follows. Section 2 reviews selected results in the literature related to POPs; Section 3 provides a complete solution for SPOPs built on a *flat poset*[3] without repeated letters. In particular, as a corollary to a more general result, we provide the generating function for the distribution of peaks (valleys) in permutations, which seems to be a new result, or at least one the author could not find in the literature (it looks like only a continued fraction expansion of the generating function for the distribution of peaks is known). Section 4 gives a q-analogue for a result on non-overlapping patterns [17, Theorem 16]. Finally, in Section 5, we state several open problems on POPs.

In what follows we need the following notations. Let σ and τ be two POPs of length greater than 0. We write $\sigma<\tau$ to indicate that any letter of σ is less than any letter of τ. We write $\sigma<>\tau$ when no letter in σ is comparable with any letter in τ. Also, segmented POP is abbreviated as *SPOP*.

A *left-to-right minimum* of a permutation π is an element a_i such that $a_i<a_j$ for every $j<i$. Analogously we define *right-to-left minimum*, *right-to-left maximum*, and *left-to-right maximum*. If $\pi=a_1a_2\cdots a_n\in\mathscr{S}_n$, then the *reverse* of π is $\pi^{\mathrm{r}}:=a_n\cdots a_2a_1$, and the *complement* of π is a permutation π^{c} such that $\pi^{\mathrm{c}}_i=n+1-a_i$, where $i\in[n]=\{1,\ldots,n\}$. We call π^{r}, π^{c}, and $(\pi^{\mathrm{r}})^{\mathrm{c}}=(\pi^{\mathrm{c}})^{\mathrm{r}}$ *trivial bijections*. The *GF* (*EGF*; *BGF*) denotes the (*exponential*; *bivariate*) *generating function*.

[2] The POPs in this paper are the same as the POGPs in [16], which is an abbreviation for partially ordered generalized patterns.

[3] The concept of a "flat poset" is used in theoretical computer science [1] to denote posets with one element being less than any other element (there are no other relations between the elements). See Fig. 5 for the shape of such poset.