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# Application of an optimization problem in Max-Plus algebra to scheduling problems

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**Abstract**

The problem tackled in this paper deals with products of a finite number of triangular matrices in Max-Plus algebra, and more precisely with an optimization problem related to the product order. We propose a polynomial time optimization algorithm for $2 \times 2$ matrices products. We show that the problem under consideration generalizes numerous scheduling problems, like single machine problems or two-machine flow shop problems. Then, we show that for $3 \times 3$ matrices, the problem is NP-hard and we propose a branch-and-bound algorithm, lower bounds and upper bounds to solve it. We show that an important number of results in the literature can be obtained by solving the presented problem, which is a generalization of single machine problems, two- and three-machine flow shop scheduling problems. The branch-and-bound algorithm is tested in the general case and for a particular case and some computational experiments are presented and discussed.


*Keywords:* Scheduling; Optimization; Max-Plus algebra

## 1. Introduction

The Max-Plus algebra has been presented by Gondran and Minoux for the computation of longest paths of valued graphs [10], among others [7]. This algebra has a lot of applications to system theory and optimal control [6,20], graph theory [10], Petri nets [8], etc. To our knowledge, just a few papers concern the application of another algebra to scheduling theory [9], and the application of this algebra to scheduling problems (see for instance [5,8,12,16]) has not been extensively studied. In this paper, we use the dioid of Max-Plus matrices and we address the problem to minimize a product of triangular matrices.

A very large literature concerns scheduling problems since the preliminary works of Johnson [14]. Several books already present general survey for these problems as [1,2,21] for the more recent ones. We show in this paper that numerous scheduling problems can be covered by a unique problem in Max-Plus algebra and one of the aims of this paper is to show that Max-Plus algebra is more adapted for solving sequencing problems than the classical $(\mathbb{R}, +, \times)$ algebra.

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We suppose in the following that we have a set of $n$ jobs $\{J_x\}_{1\leqslant x\leqslant n}$ to schedule on $m$ machines ($m$= 1, 2 or 3 in the following). Each job $J_x$ is made up of $m$ operations and the operation number $k$ is denoted by $O_{x,k}$. The processing time of operation $O_{x,k}$ is equal to $p_{x,k}$. Preemption is not allowed and a machine can only perform one job at a time.

The paper is organized as follows: in Section 2, we present Max-Plus algebra. In Section 3 we present a general optimization problem in Max-Plus algebra, that involves $2 \times 2$ matrices, and its applications to scheduling problems. In Section 4, we present this optimization problem with $3 \times 3$ matrices, we show that the problem is NP-hard and we propose lower bounds, upper bounds and a general branch-and-bound algorithm to solve it. We present some applications to scheduling problems. This branch-and-bound algorithm is tested for the general problem and for the three-machine flow shop problem in Section 5 and some computational results are discussed.

## 2. Presentation of Max-Plus algebra

In Max-Plus algebra, we denote the maximum by $\oplus$ and the addition by $\otimes$. The first operator, $\oplus$, is idempotent, commutative, associative and has a neutral element $(-\infty)$ denoted by $\mathbb{0}$. The second operator, $\otimes$, is associative, distributive on $\oplus$ and has a neutral element (0) denoted by $\mathbb{1}$. $\mathbb{0}$ is an absorbing element for $\otimes$. These properties can be summarized by saying that $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ is a dioid. It is important to note that in Max-Plus algebra and more generally in dioids, the first operator does not allow simplification: $a\oplus b = a\oplus c \not\Rightarrow b = c$. Furthermore, in $\mathbb{R}_{\max}$, the second operator $\otimes$ is commutative, and except $\mathbb{0}$, every element is invertible: the inverse of $x$ is denoted by $x^{-1}$ or $\mathbb{1}/x$. For more convenience, we denote the ordinary subtraction by $x/y$ instead of $x \otimes y^{-1}$ and by $xy$ the product $x \otimes y$.

It is possible to extend these two operators to $m \times m$ matrices of elements of $\mathbb{R}_{\max}$. Let $A$ and $B$ be two matrices of size $m \times m$, operators $\oplus$ and $\otimes$ are defined by

$$\forall (i, j) \in \{1, \ldots, m\}^2 \quad [A \oplus B]_{i,j} = A_{i,j} \oplus B_{i,j},$$

$$\forall (i, j) \in \{1, \ldots, m\}^2 \quad [A \otimes B]_{i,j} = \bigoplus_{k=1}^{m} A_{i,k} \otimes B_{k,j}.$$

It is not difficult to show that $\mathscr{M}_{m\times m}(\mathbb{R}_{\max})$, the set of $m \times m$ matrices in $\mathbb{R}_{\max}$ is a dioid. Considering a triangular matrix $A$ of size $m \times m$, such that $A_{i,j} = \mathbb{0}\ \forall i > j$, the set of $m \times m$ triangular matrices, denoted by $\mathscr{T}_{m\times m}(\mathbb{R}_{\max})$, is a dioid as well, but $\otimes$ is not commutative and not every element is invertible. More details can be found in [11].

## 3. Optimization problem in $\mathscr{T}_{2\times 2}(\mathbb{R}_{\max})$

### 3.1. Minimization of the product of matrices

Let us consider a set of $n$ $2 \times 2$ triangular matrices in Max-Plus algebra $\mathscr{M} = \{M(1), M(2), \ldots, M(n)\}$, with $\forall i, 1\leqslant i\leqslant n$:

$$M(i) = \begin{pmatrix} \mu_1(i) & \mu_{1,2}(i) \\ \mathbb{0} & \mu_2(i) \end{pmatrix}.$$

It follows from the definition that the product of two matrices $M(i)$ and $M(i')$ is equal to:

$$M(i) \otimes M(i') = \begin{pmatrix} \mu_1(i)\mu_1(i') & \mu_1(i)\mu_{1,2}(i')\oplus\mu_{1,2}(i)\mu_2(i') \\ \mathbb{0} & \mu_2(i)\mu_2(i') \end{pmatrix}.$$

It is clear that $M(i) \otimes M(i') \neq M(i') \otimes M(i)$: the product of matrices is not commutative. Furthermore, in the matrix equal to the product of these two matrices, only the top-right term depends on the order of the product.

**Proposition 1.** *Given two* $2 \times 2$ *triangular matrices A and B, defined by*

$$A = \begin{pmatrix} a_1 & a_{1,2} \\ \mathbb{0} & a_2 \end{pmatrix} \quad B = \begin{pmatrix} b_1 & b_{1,2} \\ \mathbb{0} & b_2 \end{pmatrix}.$$