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Some basic properties of multiple Hamiltonian covers

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Abstract

The aim of this work is to introduce the concept of a *multiple Hamiltonian cover* ($\mathscr{MHC}$). For the most part, attention is restricted to the class of cubic three-connected planar graphs. For those graphs having an $\mathscr{MHC}$ composed of three Hamiltonian cycles we are able to derive a Grinberg type result. On the other hand, for those graphs having an $\mathscr{MHC}$ consisting of six Hamiltonian cycles we find it convenient to impose the additional notion of *balance*, which then allows us to deduce some interesting consequences. We conclude with a problem from three-dimensional geometry. $\mathscr{MHC}$'s play a significant role in its solution.

Keywords: Cubic graphs; Covers; Hamiltonian cycles; Matchings; Dihedral angle-sum

1. Introduction

The notion of a *multiple Hamiltonian cover*, or $\mathscr{MHC}$ for short, is based on the concepts of *Hamilton decompositions* (see [2]), *cycle double covers* (see [4,17, p. 312, 19, p. 31]) and, is indeed, more closely related to the idea of *strong Hamiltonicity* introduced by Kotzig in [13] (see also Definition 3 later on).

Definition 1. A *multiple Hamiltonian cover* for a graph $\mathscr{G}$ is a collection $\mathscr{H}$ of Hamiltonian cycles of $\mathscr{G}$ with the property that each edge of $\mathscr{G}$ belongs to exactly the same fixed number of elements of $\mathscr{H}$.

If this number is k, we refer to the cover as a Hamiltonian k-cover. When $k=1$, this is simply a Hamilton decomposition. A 2-cover will be called a double cover and a 4-cover a quadruple cover.

Remark 2. A graph with an $\mathscr{MHC}$ has to be connected and vertex regular.

But where, exactly, do we find graphs with $\mathscr{MHC}$'s? The strongly Hamiltonian (or perfectly 1-factorable) graphs are one rich source:

Definition 3 (*Kotzig [13]*). A regular graph $\mathscr{G}$ is *strongly Hamiltonian* if it can be decomposed into linear factors such that the union of each two of them is a Hamilton cycle of the graph.

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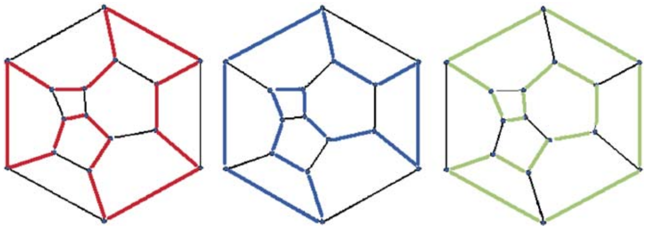

Fig. 1. Graph number 227 from Royle.

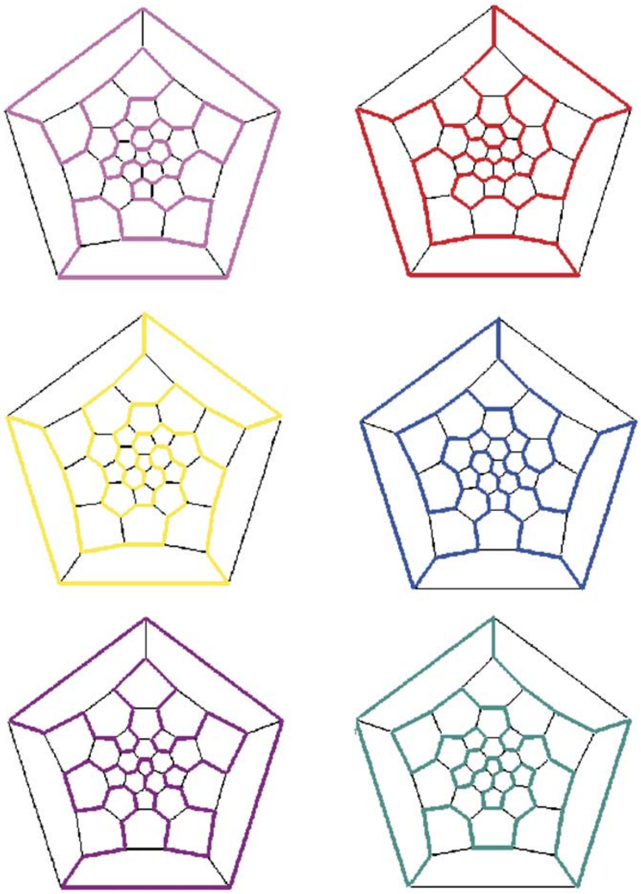

Fig. 2. Truncated icosahedron.

It should be evident from the above definition that a strongly Hamiltonian graph $\mathscr{G}$ of degree n will naturally induce an $(n-1)$-cover consisting of $\binom{n}{2}$ Hamiltonian cycles. Also, a cubic graph has a Hamiltonian double cover if and only if it is strongly Hamiltonian.

A moment's reflection also reveals that a strongly Hamiltonian graph gives rise to a Hamilton decomposition when the degree is even and to a Hamiltonian double cover when the degree is odd.

To illustrate the variety of circumstances under which we get graphs having $\mathscr{MHC}$s we have decided to include several examples: a 2-cover for our favorite graph, number 227 from Royle [15] (see Fig. 1), a 4-cover for the truncated icosahedron $5 \cdot 6^2$ (see Fig. 2), a 2-cover for the Grünbaum–Malkevitch graph [10] (see Fig. 3), a 2-cover for the icosahedron (see Fig. 4) and finally a Hamilton decomposition for the 24-cell (see Fig. 5).