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λ-Coloring matrogenic graphs[☆]

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Abstract

This paper investigates a variant of the general problem of assigning channels to the stations of a wireless network when the graph representing the possible interferences is a matrogenic graph. In our problem, channels assigned to adjacent vertices must be at least 2 apart, while channels assigned to vertices at distance 2 must be different. An exact linear time algorithm is provided for the class of threshold graphs. For matrogenic and matroidal graphs approximate algorithms are given. Consequently, previously known results concerning subclasses of cographs, split graphs and graphs with diameter 2 are improved.

Keywords: λ-Coloring; Threshold graphs; Matrogenic graphs; Matroidal graphs; Split graphs; Cographs

1. Introduction

Graph coloring is one of the most fertile and widely studied areas in graph theory, as is evident from browsing through the list of solved and unsolved problems in a comprehensive book on graph coloring problems [15]. The most general problem in this field is vertex coloring, consisting in assigning values (colors) to the vertices of a graph so that adjacent vertices have distinct colors; the objective is to minimize the number of colors used. The decision version of this problem, and of most of its modifications and generalizations, is NP-complete [11]. Generalizations of graph coloring arise in the design of wireless communication systems [13], where radio channels must be assigned to transmitters. The graph has a vertex for each transmitter and two vertices are joined by an edge if assigning them channels which are too close together could cause interference. The variant of the vertex coloring problem we focus on in this paper, the *λ-coloring problem*, consists in an assignment of colors from the integer set $0, \ldots, \lambda$ to the vertices of a graph G such that vertices at distance at most 2 get different colors and adjacent vertices get colors which are at least 2 apart. The aim is to minimize λ.

For some special classes of graphs—such as paths, cycles, wheels, tilings and k-partite graphs—tight bounds for the number of colors necessary for a λ-coloring are known and such a coloring can be computed efficiently [1,6,7,13]. Nevertheless, in general, both determining the minimum number of necessary colors [13] and deciding if this number is $< k$ for any fixed $k \geqslant 4$ [9] is NP-complete. Therefore, for many classes of graphs—such as chordal graphs [19],

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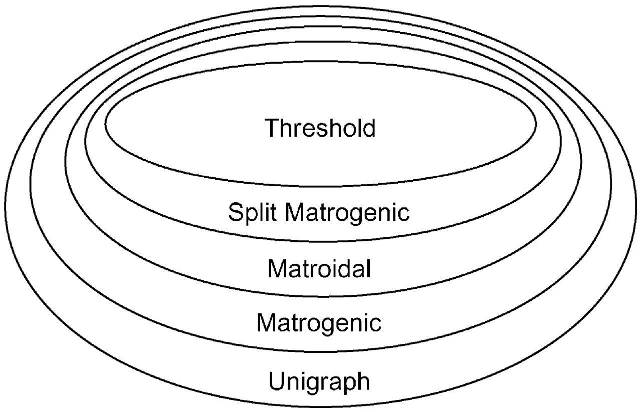

Fig. 1. Relationships of inclusion among the subclasses of unigraphs.

interval graphs [7], split graphs [2], outerplanar and planar graphs [2,6]—approximate bounds have been looked for. For a complete survey, see [4].

In this paper we consider the λ-coloring problem restricted to some subclasses of *unigraphs*, i.e. graphs uniquely determined by their own degree sequence up to isomorphism.

In [3] unigraphs are presented as a superclass including *matrogenic graphs*, *matroidal graphs*, *split matrogenic graphs* and *threshold graphs* as shown in Fig. 1 (see Section 2 for definitions). We will focus on each of these subclasses, while it remains an open problem to extend our algorithm paradigm to the whole class of unigraphs.

In the following we present linear time algorithms to λ-color these graphs, taking advantage of the degree sequence's analysis. Namely, first a general algorithm paradigm is provided, and then it is specialized for threshold, split matrogenic, matrogenic and matroidal graphs. For threshold graphs the algorithm is exact, while for the other ones it approximates the optimal solution. These algorithms improve results presented in the literature.

In particular, threshold graphs are a subclass of *cographs*, i.e. graphs not containing P_4 as induced subgraph. For cographs Chang and Kuo [7] proved that the λ-coloring problem is polynomially solvable, but they do not provide an explicit algorithm; our algorithm presents a simple method for generating an optimal solution.

Moreover, we prove some upper bounds for λ that are linear in Δ, the maximum degree of the graph. Then the upper bound $\lambda \leqslant \Delta^{1.5} + 2\Delta + 2$, shown by Bodlaender et al. in [2] for split graphs, is improved when the problem is restricted both to threshold and to split matrogenic graphs.

Finally, Griggs and Yeh [13] showed that graphs with diameter 2 satisfy $\lambda \leqslant \Delta^2$; threshold graphs have diameter 2, nevertheless for them our upper bound on λ is linear in Δ.

This paper is organized as follows. In the next section we give definitions of the classes of graphs considered in the present work and summarized in Fig. 1; then we present some properties useful in the successive sections.

In Section 3 a linear time approximation algorithm paradigm for λ-coloring matrogenic graphs is presented and its correctness is shown. This algorithm is specialized to subclasses of matrogenic graphs in the next sections. Namely, the proofs that this algorithm—when executed on threshold graphs—uses a linear number of colors in Δ and that this number is minimum are provided in Section 4. Section 5 contains the details of the algorithm for split matrogenic graphs and provides its performances. Finally, Section 6 is devoted to λ-coloring matrogenic and matroidal graphs.

2. Graph theory preliminaries

We consider only finite, simple, loopless graphs $G = (V, E)$, where V is the vertex set of G with cardinality n and E is the edge set of G with cardinality m.

A vertex $x \in V$ is called *universal* (*isolated*) if it is adjacent to all other vertices of V (no other vertex in V); if x is a universal (isolated) vertex, then its degree is $d(x) = n - 1$ $(d(x) = 0)$.

A graph $I = (V_I, E_I)$, where $V_I \subseteq V$ and $E_I = E \cap (V_I \times V_I)$ is said to be *induced* by V_I.