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Hamiltonicity and colorings of arrangement graphs

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Abstract

We study connectivity, Hamilton path and Hamilton cycle decomposition, 4-edge and 3-vertex coloring for geometric graphs arising from pseudoline (affine or projective) and pseudocircle (spherical) arrangements. While arrangements as geometric objects are well studied in discrete and computational geometry, their graph theoretical properties seem to have received little attention so far. In this paper we show that they provide well-structured examples of families of planar and projective-planar graphs with very interesting properties. Most prominently, spherical arrangements admit decompositions into two Hamilton cycles; this is a new addition to the relatively few families of 4-regular graphs that are known to have Hamiltonian decompositions. Other classes of arrangements have interesting properties as well: 4-connectivity, 3-vertex coloring or Hamilton paths and cycles. We show a number of negative results as well: there are projective arrangements which cannot be 3-vertex colored. A number of conjectures and open questions accompany our results.

Keywords: Line and pseudoline arrangement; Circle and pseudocircle arrangement; Hamilton path; Hamilton cycle; Hamilton decomposition; Coloring; Connectivity; Planar graph; Projective-planar graph

1. Introduction

We study connectivity, vertex and edge coloring and Hamiltonicity properties for classes of geometric graphs arising from finite collections of pseudolines (resp., pseudocircles) in the Euclidean and Projective planes or on the sphere **S**. Our objects of study, known as arrangement graphs in the computational or discrete geometry literature, are 4-regular and planar (or projective-planar). They arise in connection with many combinatorial or algorithmic questions involving finite sets of planar lines or (via polar-duality) points (see [7]).

We proceed to a systematic study of these properties and report a number of positive and negative results, as well as a few still open questions which resisted our methods. Our most striking result, described in Section 3, is the existence of two Hamilton path (2HP) and cycle (2HC) decompositions for spherical arrangements, obtained via a short and easy to describe construction based on wiring diagrams.

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Finding Hamilton paths and cycles in graphs is an NP-hard problem, even for planar graphs, and even for arrangement graphs of Jordan curves (see [15]). It is known that 4-connected planar graphs always have a Hamilton cycle ([26], see also [25,22]). The same property holds for 4-connected projective-planar graphs [24]. It is therefore interesting to see if the Hamilton cycles could be explicitly constructed for particular classes of graphs. We have such a simple construction for spherical arrangements and odd projective arrangements.

2HP and 2HC decompositions for 4-regular graphs have been widely studied in the graph theory community [4], but there are relatively few positive results. For instance, Hamiltonian decompositions are known to exist in 4-regular Cayley graphs [1], and also in line graphs of Hamiltonian cubic graphs [19]. Our pseudocircle and separating-circle arrangement graphs provide new significant examples.

Coloring vertices of planar graphs with few (3 or 4) colors is known either via the Four Color Theorem or for particular classes of planar graphs (such as 3-colorability of outerplanar graphs and triangle-free planar graphs). 4-edge colorability of 4-regular planar graphs arising from arrangements of planar curves is known only for special cases. There are some graph theoretical conjectures (see [16]) about 4-edge colorings of certain circle arrangements: a simple proof of them would imply a simple proof for the Four Color Theorem (see also [17, p. 45]). Although our 4-edge coloring result for spherical arrangement graphs does not seem to lead in the direction of Jaeger and Shank's conjecture, some ideas might prove relevant.

The paper is organized as follows. In Section 2 we present the definitions, preliminaries and basic results on connectivity, coloring and Hamiltonicity pertaining to our three geometric models: projective, Euclidean and spherical. In Section 3, we present the wiring diagram technique for constructing Hamilton path and cycle decompositions for spherical arrangements and partial results in the projective setting. Open problems and conjectures follow the natural flow of the paper.

Arrangement graphs are defined precisely in the next section. We remark that there is no connection between our arrangement graphs and another family of graphs that bears the same name and which play a role in the field of network design [6,21].

2. Arrangement graphs: preliminaries

The general objects of our study are arrangement graphs arising from finite sets of curves obeying specific intersection rules and which live in the Euclidean or projective plane or on the two-dimensional sphere. In this section, we introduce three classes of arrangements and their corresponding arrangement graphs. We illustrate the definitions by examples and provide proofs of some elementary structural properties concerning connectivity and coloring.

2.1. Projective lines

Arrangements of straight lines are among the most basic objects one may study in the real projective plane **P**. Accordingly they have been and still are studied under a vast variety of aspects. See the overviews by Grünbaum [13] and Erdős and Purdy [8] for further pointers to the field. Many combinatorial properties of arrangements of lines do not depend on the fact that the lines are straight, but rather on the nature of their incidence properties. This leads to the natural generalization, first done by Levi [20], to arrangements of pseudolines. See [11] for a comprehensive survey.

An *arrangement of pseudolines* in the projective plane **P** is a family of simple closed curves (called *pseudolines*) such that every two curves have exactly one point in common, where they cross. If no point belongs to more than two of the (pseudo) lines the arrangement is called *simple*, otherwise it is *non-simple*.

Pseudoline arrangements provide generic models for the (purely combinatorial) oriented matroids of rank 3 (see [2]). In this paper we will work only with this model. A few simplifying assumptions: we will work only with simple arrangements. We also simplify the terminology by dropping the *pseudo* prefix from *pseudoline*: all the results of this paper hold in this more general context, and straightness of lines is no issue.

With an arrangement we associate the cell complex of vertices, edges and two-dimensional regions into which the lines of the arrangement decompose the underlying space **P**. Arrangements are *isomorphic* provided their cell complexes are isomorphic. A projective arrangement graph is the graph of vertices and edges of an arrangement of pseudolines. See Fig. 1 for an example.