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Communication

Minimum cost and list homomorphisms to semicomplete digraphs

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Abstract

For digraphs D and H, a mapping $f : V(D) \to V(H)$ is a *homomorphism of D to H* if $uv \in A(D)$ implies $f(u)f(v) \in A(H)$. Let H be a fixed directed or undirected graph. The *homomorphism problem* for H asks whether a directed or undirected input graph D admits a homomorphism to H. The *list homomorphism problem* for H is a generalization of the homomorphism problem for H, where every vertex $x \in V(D)$ is assigned a set L_x of possible colors (vertices of H).

The following optimization version of these decision problems generalizes the list homomorphism problem and was introduced in Gutin et al. [Level of repair analysis and minimum cost homomorphisms of graphs, Discrete Appl. Math., to appear], where it was motivated by a real-world problem in defence logistics. Suppose we are given a pair of digraphs D, H and a positive integral cost $c_i(u)$ for each $u \in V(D)$ and $i \in V(H)$. The cost of a homomorphism f of D to H is $\sum_{u\in V(D)} c_{f(u)}(u)$. For a fixed digraph H, the *minimum cost homomorphism problem* for H is stated as follows: for an input digraph D and costs $c_i(u)$ for each $u \in V(D)$ and $i \in V(H)$, verify whether there is a homomorphism of D to H and, if one exists, find such a homomorphism of minimum cost.

We obtain dichotomy classifications of the computational complexity of the list homomorphism and minimum cost homomorphism problems, when H is a semicomplete digraph (digraph in which there is at least one arc between any two vertices). Our dichotomy for the list homomorphism problem coincides with the one obtained by Bang-Jensen, Hell and MacGillivray in 1988 for the homomorphism problem when H is a semicomplete digraph: both problems are polynomial solvable if H has at most one cycle; otherwise, both problems are NP-complete. The dichotomy for the minimum cost homomorphism problem is different: the problem is polynomial time solvable if H is acyclic or H is a cycle of length 2 or 3; otherwise, the problem is NP-hard.

Keywords: Homomorphisms; Digraphs; Semicomplete digraphs

1. Introduction

For excellent introductions to homomorphisms in directed and undirected graphs, see [20,22]. In this paper, directed (undirected) graphs have no parallel arcs (edges) or loops. The vertex (arc) set of a digraph G is denoted by $V(G)$ ($A(G)$). The vertex (edge) set of an undirected graph G is denoted by $V(G)$ ($E(G)$). For a digraph G, if $xy \in A(G)$, we say that x *dominates* y and y is *dominated* by x. A *k-cycle*, denoted by $\vec{C}_k$, is a directed simple cycle with k vertices. A digraph is *acyclic* if it has no cycle. A digraph D is *semicomplete* if, for each pair x, y of distinct vertices either x

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dominates y or y dominates x or both. A *tournament* is a semicomplete digraph with no 2-cycle. Semicomplete digraphs and, in particular, tournaments are well-studied in graph theory and algorithms [4]. A digraph G' is the *dual* of a digraph G if G' is obtained from G by reversing the orientation of all arcs.

For digraphs D and H, a mapping $f : V(D) \to V(H)$ is a *homomorphism of D to H* if $uv \in A(D)$ implies $f(u)f(v) \in A(H)$. A homomorphism f of D to H is also called an *H-coloring* of D, and $f(x)$ is called the *color* of the vertex x in D. We denote the set of all homomorphisms from D to H by $HOM(D, H)$. Let H be a fixed digraph. The *homomorphism problem for H*, HOMP(H), asks whether there is a homomorphism of an input digraph D to H (i.e., whether $HOM(D, H) \neq \emptyset$). In the *list homomorphism problem for H*, LHOMP(H), we given an input digraph D and a set (called a *list*) $L_v \subseteq V(H)$ for each $v \in V(D)$. Our aim is to check whether there is a homomorphism $f \in HOM(D, H)$ such that $f(v) \in L_v$ for each $v \in V(D)$.

The problems HOMP(H) and LHOMP(H) have been studied for several families of directed and undirected graphs H, see, e.g., [20,22]. A well-known result of Hell and Nešetřil [21] asserts that HOMP(H) for undirected graphs is polynomial time solvable if H is bipartite and it is NP-complete, otherwise. Feder et al. [11] proved that LHOMP(H) for undirected graphs is polynomial time solvable if H is a bipartite graph whose complement is a circular arc graph (a graph isomorphic to the intersection graph of arcs on a circle), and LHOMP(H) is NP-complete, otherwise. Such a dichotomy classification is not known for the homomorphism problems HOMP(H) when H is a digraph and only partial classifications have been obtained, see [22]. For example, Bang-Jensen et al. [5] showed that HOMP(H) for semicomplete digraphs H is polynomial time solvable if H has at most one cycle and HOMP(H) is NP-complete, otherwise. Nevertheless, Bulatov [7] managed to prove that each list homomorphism problem LHOMP(H) is polynomial time solvable or NP-complete. Such a dichotomy result for HOMP(H) has been conjectured, see, e.g., [20,22]. If this conjecture holds, it will imply that the well-known Constraint Satisfaction Problem Dichotomy Conjecture of Feder and Vardi also holds [12].

The authors of [16] introduced an optimization problem, MinHOMP(H), on H-colorings of undirected graphs H. The problem is motivated by a problem in defence logistics. Suppose we are given a pair of digraphs D, H and a positive integral cost $c_i(u)$ for each $u \in V(D)$ and $i \in V(H)$. The *cost* of a homomorphism f of D to H is $\sum_{u \in V(D)} c_{f(u)}(u)$. For a fixed digraph H, the *minimum cost homomorphism problem* MinHOMP(H) is stated as follows: for an input digraph D and costs $c_i(u)$ for each $u \in V(D)$ and $i \in V(H)$, verify whether $HOM(D, H) \neq \emptyset$ and, if $HOM(D, H) \neq \emptyset$, find a homomorphism in $HOM(D, H)$ of minimum cost. The problem MinHOMP(H) generalizes LHOMP(H) (and, thus, HOMP(H)): assign $c_i(u) = 1$ if $i \in L_u$ and $c_i(u) = 2$, otherwise. Then a list homomorphism with respect to lists L_u, $u \in V(D)$, exists if and only if there is a homomorphism of D to H of cost $|V(D)|$.

In this paper, we obtain dichotomy classifications for the time complexity of LHOMP(H) and MinHOMP(H) when H is a semicomplete digraph. Our classification for LHOMP(H) coincides with that for HOMP(H) [5] described earlier. However, for MinHOMP(H) the classification is different: the problem is polynomial time solvable when H is either an acyclic tournament or a 2-cycle or a 3-cycle. Otherwise, MinHOMP(H) is NP-hard. This implies that even when H is a unicyclic semicomplete digraph on at least four vertices, MinHOMP(H) is NP-hard (unlike HOMP(H) and LHOMP(H)).

Cohen et al. [8,9] considered an optimization version of the well-known constraint satisfaction problem (CSP), the valued CSP (VCSP). Special cases of VCSP were studied in several other papers including [10], where weighted Max CSP is investigated. The problem VCSP and some of its special cases generalize MinHOMP(H). We consider VCSP in the next section and demonstrate that an important result on VCSP describing some polynomial cases can be applied to MinHOMP(H). However, since VCSP is a proper generalization of MinHOMP(H) we could not possibly use NP-hardness results proved for VCSP. Moreover, many of these NP-hardness results are for some special cases of VCSP that do not generalize MinHOMP(H).

VCSP extends another optimization problem on H-colorings, the minimum graph homomorphism problem, introduced in [1]. However, the authors of [1] considered only reflexive undirected graphs H, i.e., graphs in which every vertex of H has a loop, and the costs are assigned only to edges of H. Thus, MinHOMP(H) and the minimum graph homomorphism problem from [1] are rather different problems. Another related but different homomorphism problem on weighted graphs is investigated in [15].

The *maximum cost homomorphism problem* MaxHOMP(H) is the same problem as MinHOMP(H), but instead of minimization we consider maximization. Let M be a constant larger than any cost $c_i(u)$, $u \in V(D)$, $i \in V(H)$. Then the cost $c'_i(u) = M - c_i(u)$ is positive for each $u \in V(D)$, $i \in V(H)$. Due to this transformation, the problems MinHOMP(H) and MaxHOMP(H) are equivalent. Note that allowing negative or zero costs would not make MinHOMP(H) and