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On the non-unit count of interval graphs

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ABSTRACT

We introduce the non-unit count of an interval graph as the minimum number of intervals in an interval representation whose lengths deviate from one. We characterize a variant of the non-unit count (where all interval lengths are required to be at least one) and graphs with non-unit count 1.

1. Introduction

Interval graphs reflect the intersection structure of intervals in the real line. For each vertex of an interval graph $G = (V, E)$ there is an interval I_v such that $uv \in E$ if and only if $I_u \cap I_v \neq \emptyset$. Each such collection also defines an *interval order* as a partial order $P = (V, <)$ via $u < v$ if and only if I_u is completely left of I_v. Interval graphs and interval orders have been characterized in various ways (cf., e.g. [6,8,11]). For more details, the interested reader is referred to [5,9].

A natural question, apparently first asked by R.L. Graham, is how many different interval lengths are needed to represent an interval graph. He introduced the *interval count* as the minimum number of distinct interval lengths necessary in an interval representation of an interval graph. Interval graphs with interval count 1 are the *unit interval graphs*. They were first characterized by Roberts [15] as the class of proper interval graphs or, equivalently, as the claw-free interval graphs. Shorter proofs of these characterizations and efficient recognition algorithms can be found in, e.g., [1,4,7,12]. For interval graphs with interval count k with $k \geq 2$ the recognition problem seems to be open. Further results on the interval count can be found in e.g. [2,3].

Graham conjectured that the interval count of a graph G is at most $k + 1$ if for some vertex x the graph $G \setminus x$ has interval count k. This conjecture was proved by Leibowitz et al. [10] for $k = 1$ and disproved for $k \geq 2$. Observe that in the first case, $G \setminus x$ is a unit interval graph and x must be contained in every claw of G.

Skrien [16] and Rautenbach and Szwarcfiter [13] investigated a subclass of graphs with interval count 2. In [13] Rautenbach and Szwarcfiter give a forbidden subgraph characterization of those graphs that have an interval representation using unit intervals and single points. They also describe a linear time recognition algorithm. In [14] the same authors characterize graphs having a representation by open and closed unit intervals.

In the following we ask a slightly different question: how many intervals in an interval representation must have a length different from one? In Section 2 we collect the basic notations and definitions. In Sections 3 and 4 we present some general results and give exact answers for two special cases where all interval lengths are required to be at least one or where all but one interval have the same length.

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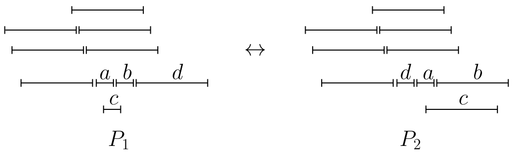

Fig. 1. Two equivalent interval orders with different non-unit count.

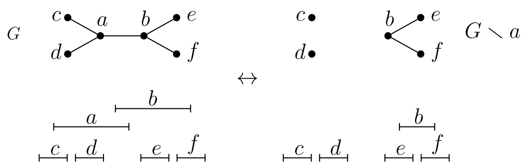

Fig. 2. $G \setminus a$ is claw-free but $\tau(G) = 2$.

2. Preliminaries

Let $G = (V, E)$ be an interval graph and R a collection of intervals such that for each $v \in V$ there is an interval $I_v \in R$ and $uv \in E$ if and only if I_u and I_v have a nonempty intersection. We then say that R *realizes* G. The set of all collections of intervals realizing G is denoted by $\mathcal{R}(G)$. For an interval I let $l(I)$ $(r(I))$ be its left (right) endpoint. W.l.o.g. we assume throughout that all interval endpoints are distinct. Let $|I|$ denote the length of I. The collection R of intervals also realizes the partial order $(V, <)$ given by $u < v$ if and only if $r(I_u) < l(I_v)$. $\mathcal{R}(P)$ is the set of all realizers of the interval order P.

We call two interval orders P_1 and P_2 equivalent ($P_1 \sim P_2$) if they have realizers which realize the same interval graph G. The corresponding equivalence class is denoted by $\mathcal{P}(G)$. A function f operating on interval orders is a *comparability invariant* if $f(P_1) = f(P_2)$ whenever $P_1 \sim P_2$.

We call a bipartite graph $K_{1,r}$, $r \geq 3$, a *star* and a *claw*, if $r = 3$. A vertex u of a graph G is a *star center* of G if u together with some of its neighbors $N'(u) \subseteq N(u)$ induces a star in G. The vertices in $N'(u)$ are then called *leaves*. The set of star centers of G is denoted by $Z = Z(G)$, the set of leaves by $U = U(G)$. Observe that the sets Z and U do not have to be disjoint. For an interval order P the star centers $Z(P)$ of P are the star centers of the corresponding interval graph.

An interval graph is a *unit interval graph* if it has a realizer in which all intervals have the same length. The corresponding partial order is then called a *semiorder*. We call a collection of intervals R a *proper collection* if in R no interval properly contains another. Roberts [15] showed that unit interval graphs are characterized by the fact that they have a realization by a proper collection of intervals. Moreover, they are precisely the interval graphs having no induced *claw*.

Let P be an interval order and $R \in \mathcal{R}(P)$ a realization. The *non-unit count* $\tau(R)$ of R is the number of intervals in R with length different from one. Similarly, let

$$\tau(P) = \min\{\tau(R) | R \in \mathcal{R}(P)\} \tag{1}$$

and

$$\tau(G) = \min\{\tau(R) | R \in \mathcal{R}(G)\}. \tag{2}$$

Then, for an interval graph G and $P \in \mathcal{P}(G)$ we have $\tau(G) \leq \tau(P)$ and, obviously, $\tau(G) = \min\{\tau(P) | P \in \mathcal{P}(G)\}$. For unit interval graphs G we have $\tau(G) = \tau(P) = 0$ for all semiorders $P \in \mathcal{P}(G)$. This seems to suggest that the non-unit count is a comparability invariant. This, however, is not true. Fig. 1 shows two interval orders with $P_1 \sim P_2$ and $\tau(P_1) = 3$, $\tau(P_2) = 2$.

Since unit interval graphs are the claw-free interval graphs, one may conjecture that the non-unit count is the cardinality $\nu(G)$ of the smallest set $W \subseteq V$ such that $G \setminus W$ is claw-free. Again, this is not true. To see this, consider Fig. 2.

If we remove either a or b, the resulting graph is claw-free. On the other hand, the intervals I_a and I_b have to overlap and each of both has to cover one of two non-overlapping intervals. So $\tau(G) = 2$.

Indeed, in general we have $\nu(G) \leq \tau(G)$. A related parameter is introduced in [5]. Fishburn defines $\kappa(n)$ as the maximum cardinality k such that every interval graph on n vertices contains a unit interval graph on k vertices. For interval graphs G on n vertices this obviously gives $\nu(G) \leq n - \kappa(n)$.

3. The normalized non-unit count

In this section we consider the *normalized non-unit count* where we consider only interval representations in which all intervals have length at least one. We denote the normalized non-unit count as $\tau_>(R)$ (and $\tau_>(P)$, $\tau_>(G)$, resp.).

Consider an interval representation $R \in \mathcal{R}(G)$ and a star center $z \in Z$. Then I_z properly contains at least one interval I_x for some $x \in V \setminus Z$. Hence, if $|I_x| \geq 1$, then $|I_z| > 1$. This immediately implies $\tau_>(G) \geq |Z|$. Since $\tau_>(P) \geq \tau_>(G)$ for all $P \in \mathcal{P}(G)$, we also have $\tau_>(P) \geq |Z|$.