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Best monotone degree conditions for binding number and cycle structure

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ABSTRACT

Woodall has shown that every 3/2-binding graph is hamiltonian. In this paper, we consider best monotone degree conditions for a b-binding graph to be hamiltonian, for $1 \leq b < 3/2$. We first establish such a condition for $b = 1$. We then give a best monotone degree condition for a b-binding graph to be 1-tough, for $1 < b < 3/2$, and conjecture that this condition is also the best monotone degree condition for a b-binding graph to be hamiltonian, for $1 < b < 3/2$.

1. Introduction

We consider only simple graphs without loops or multiple edges. Our terminology and notation are standard except as indicated, and a good reference for any undefined terms or notation is [12]. We mention that for two graphs G, H on disjoint vertex sets, we will denote their *disjoint union* by $G \cup H$ and their *join* by $G + H$.

An integer sequence $\pi = (d_1 \leq d_2 \leq \cdots \leq d_n)$ is called *graphical* if there exists a graph G having π as its degree sequence, and we then call G a *realization* of π. If P is a graph property (e.g., hamiltonicity, k-connectedness), we call a graphical sequence π *forcibly* P if every realization of π has property P. If $\pi = (d_1 \leq d_2 \leq \cdots \leq d_n)$ and $\pi' = (d'_1 \leq d'_2 \leq \cdots \leq d'_n)$ are two n-sequences, not necessarily graphical, we say π' *majorizes* π, denoted $\pi' \geq \pi$, if $d'_i \geq d_i$ for all i. We write $\pi' \gneq \pi$ if $\pi' \geq \pi$ and $\pi' \neq \pi$.

A condition on the vertex degrees of a graph is called *monotone* if $\pi' \geq \pi$ satisfies the condition whenever π does, and a graph property is called *ancestral* if adding edges preserves the property. Historically, monotone conditions on the vertex degrees have been used to provide sufficient conditions for a graph to have certain ancestral properties, such as k-connectedness [6,7]. A general method for constructing such conditions was given in [8].

Sufficient conditions for π to be forcibly hamiltonian were given by several authors [7,10,11], culminating in the following theorem of Chvátal [9].

Theorem 1.1. *Let $\pi = (d_1 \leq d_2 \leq \cdots \leq d_n)$ be a graphical sequence, with $n \geq 3$. If $d_i \leq i \Longrightarrow d_{n-i} \geq n - i$ for $1 \leq i < \frac{n}{2}$, then π is forcibly hamiltonian.*

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Unlike its predecessors, Theorem 1.1 has the property that if a sequence π fails to satisfy it, then π is majorized by a sequence π' with a nonhamiltonian realization G'. As we will see below, this implies that the degree condition in Theorem 1.1 is the best possible monotone condition to guarantee that π is forcibly hamiltonian. For this reason, we call Theorem 1.1 a best monotone hamiltonian theorem.

Recent work has focused almost entirely on finding best monotone degree conditions for various single ancestral graph properties (e.g., k-edge-connectedness [3], having a 2-factor [1], etc.). In this paper, we consider how to establish best monotone degree conditions to guarantee that a graph with property P_1 also has property P_2, where P_1 and P_2 are ancestral graph properties. We will refer to such conditions as best monotone $(P_1 \Rightarrow P_2)$ conditions. A formal framework to establish that $(P_1 \Rightarrow P_2)$ conditions are best monotone is given in Section 2. We will be particularly interested in the case where P_1 is 'b-binding' and P_2 is '1-tough', ancestral graph properties which we now define.

In [13], Woodall introduced the notion of the binding number of a graph G. If $S \subseteq V(G)$, let $N(S)$ denote the set of neighbors of S in G, including any vertices of S that have neighbors in S. For $b \geq 0$, we call G *b-binding* if $b|S| \leq |N(S)|$, for all $S \subseteq V(G)$ with $N(S) \neq V(G)$. The *binding number* of G, denoted $\text{bind}(G)$, is the maximum $b \geq 0$ such that G is b-binding. Thus,

$$\text{bind}(G) = \min\left\{ \frac{|N(S)|}{|S|} \,\middle|\, \emptyset \neq S \subseteq V(G),\ N(S) \neq V(G) \right\},$$

and a set $S \subseteq V(G)$ attaining this minimum is called a *binding set* of G. In particular, $\text{bind}(K_n) = n - 1$. A best monotone b-binding theorem was given in [4].

The most important theorem relating binding number and cycle structure is the following result of Woodall [13,14], in which the constant $3/2$ is best possible.

Theorem 1.2. *If* $\text{bind}(G) \geq \frac{3}{2}$*, then G is hamiltonian.*

Chvátal introduced the notion of the toughness of a graph in [9]. Let $\omega(G)$ denote the number of components of a graph G. For $t \geq 0$, we call G *t-tough* if $t \cdot \omega(G - X) \leq |X|$, for every $X \subseteq V(G)$ with $\omega(G - X) \geq 2$. The *toughness* of a noncomplete graph G, denoted $\tau(G)$, is the maximum $t \geq 0$ for which G is t-tough, so that

$$\tau(G) = \min\left\{ \frac{|X|}{\omega(G - X)} \,\middle|\, X \subset V(G) \text{ and } \omega(G - X) \geq 2 \right\}.$$

By convention, $\tau(K_n) := (n - 1)$. A best monotone t-tough theorem was given in [2].

In Sections 3 through 6, we consider best monotone (b-binding $\Rightarrow$ hamiltonian) degree conditions, for $1 \leq b \leq 3/2$. In Section 3 we give such a condition when $b = 1$. After describing a method to do so in Section 4, we construct in Section 5 a best monotone (b-binding $\Rightarrow$ 1-tough) degree condition, for $1 < b < 3/2$. Finally, in Section 6, we apply the framework of Section 2 to describe exactly what would be needed to show that the best monotone (b-binding $\Rightarrow$ 1-tough) condition in Section 5 is also a best monotone (b-binding $\Rightarrow$ hamiltonian) condition, for $1 < b < 3/2$.

2. Best monotone $(P_1 \Rightarrow P_2)$ theorems

Throughout this paper, graph properties will be assumed ancestral. If P_1 and P_2 are graph properties (e.g., being 1-tough, hamiltonian, etc.), we call a graphical sequence π *forcibly* $P_1 \Rightarrow P_2$ if every realization of π with property P_1 also has property P_2. Given graph properties P_1 and P_2, consider a theorem T which declares certain degree sequences to be forcibly $P_1 \Rightarrow P_2$, rendering no decision on the remaining degree sequences. We call such a theorem T a *forcibly* $(P_1 \Rightarrow P_2)$ *theorem* (or just a $(P_1 \Rightarrow P_2)$ *theorem*, for brevity). We call a $(P_1 \Rightarrow P_2)$ theorem T *monotone* if, for any two degree sequences π, π', whenever T declares π forcibly $P_1 \Rightarrow P_2$ and $\pi' \geq \pi$, then T declares π' forcibly $P_1 \Rightarrow P_2$. We call a $(P_1 \Rightarrow P_2)$ theorem T *optimal* if whenever T does not declare π forcibly $P_1 \Rightarrow P_2$, then π is not forcibly $P_1 \Rightarrow P_2$. We call a $(P_1 \Rightarrow P_2)$ theorem T *weakly optimal* if for any sequence π (not necessarily graphical) which T does not declare forcibly $P_1 \Rightarrow P_2$, π is majorized by a degree (i.e., graphical) sequence which is not forcibly $P_1 \Rightarrow P_2$. In view of the following result, a $(P_1 \Rightarrow P_2)$ theorem which is both monotone and weakly optimal is called a *best monotone* $(P_1 \Rightarrow P_2)$ *theorem*.

Theorem 2.1. *Let T, T_0 be monotone $(P_1 \Rightarrow P_2)$ theorems, with T_0 weakly optimal. If T declares a degree sequence π to be forcibly $P_1 \Rightarrow P_2$, then so does T_0.*

Proof. Suppose there exists a degree sequence π that T declares forcibly $P_1 \Rightarrow P_2$, but T_0 does not. Since T_0 is weakly optimal, there exists a degree sequence $\pi' \geq \pi$ which is not forcibly $P_1 \Rightarrow P_2$. This means that T does not declare π' forcibly $P_1 \Rightarrow P_2$. But if T declares π forcibly $P_1 \Rightarrow P_2$, $\pi' \geq \pi$, and T does not declare π' forcibly $P_1 \Rightarrow P_2$, then T is not monotone, a contradiction. ■

To illustrate the concepts just introduced, we present two easy examples of a best monotone $(P_1 \Rightarrow P_2)$ theorem. Both are slightly modified versions of Theorem 1.1. The first example is a best monotone (2-connected $\Rightarrow$ hamiltonian) theorem.

Theorem 2.2. *Let $\pi = (d_1 \leq d_2 \leq \cdots \leq d_n)$ be a graphical sequence. If $d_i \leq i$ implies $d_{n-i} \geq n - i$, for $2 \leq i < \frac{n}{2}$, then every 2-connected realization of π is hamiltonian.*