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Faster separation of 1-wheel inequalities by graph products

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ABSTRACT

Using graph products we present an $O(|V|^2|E| + |V|^3 \log |V|)$ separation algorithm for the nonsimple 1-wheel inequalities by Cheng and Cunningham (1997) of the stable set polytope, which is faster than their $O(|V|^4)$ algorithm.

There are two ingredients for our algorithm. The main improvement stems from a reduction of the separation problem to multiple shortest path problems in an auxiliary graph having only $6|V|$ vertices and $9|E|$ arcs, thereby preserving low sparsity. Then Johnson's algorithm can be applied to the auxiliary graph of same sparsity as the original one.

In contrast, Cheng and Cunningham's auxiliary graph is by construction dense, $|E| = O(|V|^2)$, so application of Johnson's algorithm provides no large improvement.

Many important application problems contain large subproblems of the following *binary packing* type:

$$\begin{aligned} \max\ & c^\top x \\ \text{s.t. } & Ax \le b \\ & x \in \{0,1\}^n \end{aligned} \tag{BPP}$$

where (A, b) is a matrix of nonnegative integers. In the process of solving (BPP) by a branch-and-cut algorithm, it is for the cutting-part helpful to associate with it Padberg's *conflict/intersection graph*, see [10,2]. Let $V = \{1, \dots, n\}$ and any pair of vertices i, j be adjacent if for the columns a^i and a^j hold that $a^i + a^j \not\le b$. We denote the resulting conflict graph with $G_{A,b}$. Now any integral and feasible x^* is also a feasible solution for the stable set polytope of $G_{A,b}$. Hence the stable set polytope of $G_{A,b}$ provides an integral relaxation of (BPP).

1. Introduction

Let $G = (V, E)$ be a simple connected graph with $|V| = n \ge 2$ and $|E| = m$. A subset of V is called *stable* if it does not contain adjacent vertices of G. The *incidence vector* of a set $N \subseteq V$ is $\chi^N \in \{0,1\}^V$ such that $\chi^N_v = 1$ if $v \in N$ and otherwise $\chi^N_v = 0$. The *stable set polytope* of G, denoted by STAB(G), is the convex hull of incidence vectors of stable sets of G. Some well-known valid inequalities for STAB(G) include the *trivial inequalities* ($x_v \ge 0$ for $v \in V$), the *odd cycle inequalities* ($\sum_{v\in C} x_v \le k$ where C is the vertex-set of an odd cycle of length $2k + 1$), and the *clique inequalities* ($\sum_{v\in K} x_v \le 1$ where K induces a clique). A clique inequality is called *edge inequality* if the clique has just two vertices. Let ESTAB$(G) := \{x \in [0, 1]^V : x_u + x_v \le 1\ \forall uv \in E\}$ and CSTAB$(G) := \{x \in \text{ESTAB}(G) : x$ fulfills the odd cycle inequalities$\}$.

The *separation problem* for a class $\mathcal{C}$ of valid inequalities for a class of polytopes P is: Given $x^* \in P$, does x^* violate one of the inequalities in $\mathcal{C}$? If the answer is *yes*, exhibit such an inequality. Solving this problem is important to use the inequalities

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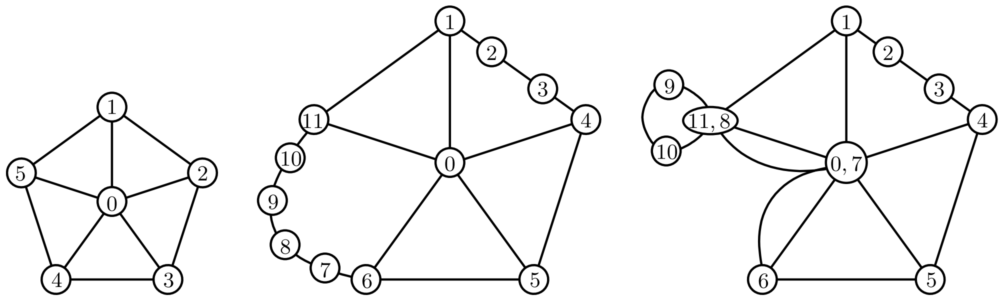

(a) wheel.

(b) simple odd $(11, 5)$-wheel $W(0; 5; 2; 1, 4, 5, 6, 11)$.

(c) nonsimple odd wheel obtained by identifying vertices $0, 7$ and $8, 11$.

Fig. 1. From a wheel to a simple odd $(11, 5)$-wheel $W(0; 5; 2; 1, 4, 5, 6, 11)$ to a nonsimple odd wheel obtained by identifying vertices $0, 7$ and $8, 11$.

in a branch-and-cut approach for maximizing a linear function over some general integer program or STAB(G). (See, for example, Barahona et al. [1] and Nemhauser and Sigismondi [9].) Furthermore, if the separation problem for $\mathcal{C}$ is solvable in polynomial time, then the linear optimization problem over $\mathcal{C}$ can be solved in polynomial time, see Grötschel et al. [6]. The separation problem for $\mathcal{C} = \{\text{trivial and edge inequalities}\}$ can obviously be solved in $O(m)$ time. If x^* satisfies the trivial and edge inequalities, then one can decide whether x^* violates an odd cycle inequality in polynomial time, as was first observed by Grötschel and Pulleyblank [7] and Grötschel et al. [6]. Odd cycle inequalities can be separated by n applications of the fast Dijkstra algorithm by Fredman and Tarjan [5] in time $O(nm + n^2 \log n)$. Hence the separation problem for the trivial, the edge, and the odd cycle inequalities can be solved in the same time.

Cheng and Cunningham [3,4] describe a way to separate the 1-wheel inequalities in time $O(n^4)$. They achieve this, by reducing the separation problem to multiple shortest path problems in dense graphs on $O(n)$ vertices.

In the present study we reduce the complexity of the separation problem of 1-wheels down to $O(n^2m + n^3 \log n)$. As stable set problems often originate from the conflict graphs of more general integer programs, and as these conflict graphs tend to be sparse, this faster algorithm is important for practical applications.

In contrast to Cheng and Cunningham's approach we construct a new auxiliary graph that is the categorical product of the original graph and a gadget on just 6 vertices; separation boils down to shortest path problems in this auxiliary graph solved by Johnson's algorithm. As the runtime of Johnson's algorithm depends on the number of edges of the auxiliary graph, which for our construction is just a constant multiple of the original number of edges, our approach is able to exploit sparsity of the original graph.

Plain application of Johnson's algorithm to Cheng and Cunningham's auxiliary graph cannot achieve a comparable speed-up, as their auxiliary graph is dense by construction.

Our approach will be to consider a least-weight wheel problem and its reduction to a shortest path problem in a product graph. Then we show that the separation problem reduces to the least-weight wheel problem.

2. Wheels

Cheng and Cunningham [3,4] consider a wheel with $(2k + 1)$ vertices and hub h (for an example of a wheel on 5 vertices and hub, see Fig. 1(a)) and its subdivisions, see Fig. 1(b). Let $1, \dots, 2k' + 1$ be the rim where the *spoke ends* are l_1 up to l_{2k+1}, ordered so that $1 = l_1 < l_2 < \cdots < l_{2k+1} \le 2k' + 1$. Denote the *spoke paths* connecting h to some l_i by P_{l_i} and their subpaths that exclude both ends by $\mathring{P}_{l_i}$. With $|\mathring{P}_{l_i}|$ we denote the number of vertices in $\mathring{P}_{l_i}$. Let the interior of the spoke paths be pairwise disjoint and let the interior of the spoke paths be disjoint to the rim. A *wheel* has to fulfill additionally the condition that the cycles $h, \mathring{P}_{l_i}, l_i, l_i + 1, \dots, l_{i+1}, \mathring{P}_{l_{i+1}}, h$ are odd for $i = 1, 2, \dots, 2k + 1$; for a complete specification we denote it by $W(h; k'; k; l_1, l_2, \dots, l_{2k+1}; P_{l_1}, P_{l_2}, \dots, P_{l_{2k+1}})$.

For a given wheel let $\mathcal{E}$ be the set of those l_i for which the paths P_{l_i} have an *even* number of edges, and let $\mathcal{O}$ be the set of remaining spoke ends. Cheng and Cunningham [3,4] show that the inequalities

$$kx_h + \sum_{i=1}^{2k'+1} x_i + \sum_{i\in\mathcal{E}} x_i + \sum_{i=1}^{2k+1} x(\mathring{P}_{l_i}) \le k' + \frac{|\mathcal{E}| + \sum_{i=1}^{2k+1} |\mathring{P}_{l_i}|}{2} \tag{$I_{\mathcal{A}}$}$$

$$(k+1)x_h + \sum_{i=1}^{2k'+1} x_i + \sum_{i\in\mathcal{O}} x_i + \sum_{i=1}^{2k+1} x(\mathring{P}_{l_i}) \le k' + \frac{|\mathcal{O}| + 1 + \sum_{i=1}^{2k+1} |\mathring{P}_{l_i}|}{2} \tag{$I_{\mathcal{B}}$}$$

are valid and they give sufficient conditions for them to induce facets. (Here we use $x(\mathring{P})$ for a walk $P = v_0 - \cdots - v_{k+1}$ as a shorthand for $\sum_{i=1}^{k} x_{v_i}$.)