The double competition hypergraph of a digraph

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ABSTRACT

In this paper, we introduce the notion of the double competition hypergraph of a digraph. We give characterizations of the double competition hypergraphs of arbitrary digraphs, loopless digraphs, reflexive digraphs, and acyclic digraphs in terms of hyperedge labelings of the hypergraphs.

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1. Introduction

A *digraph D* is a pair $(V(D), A(D))$ of a set $V(D)$ of *vertices* and a set $A(D)$ of ordered pairs of vertices, called *arcs*. An arc of the form (v, v) is called a *loop*. For a vertex x in a digraph D, we denote the *out-neighborhood* of x in D by $N_D^+(x)$ and the *in-neighborhood* of x in D by $N_D^-(x)$, i.e., $N_D^+(x) := \{v \in V(D) \mid (x, v) \in A(D)\}$ and $N_D^-(x) := \{v \in V(D) \mid (v, x) \in A(D)\}$. A *graph G* is a pair $(V(G), E(G))$ of a set $V(G)$ of *vertices* and a set $E(G)$ of unordered pairs of vertices, called *edges*. The *competition graph* of a digraph D is the graph which has the same vertex set as D and has an edge between two distinct vertices x and y if and only if $N_D^+(x) \cap N_D^+(y) \neq \emptyset$. In other words, the competition graph of a digraph is the intersection graph of the family of the out-neighborhoods of the vertices of the digraph (see [4] for intersection graphs). This notion was introduced by J.E. Cohen [1] in 1968 in connection with a problem in ecology, and several variants and generalizations of competition graphs have been studied.

In 1987, D.D. Scott [8] introduced the notion of double competition graphs as a variant of the notion of competition graphs. The *double competition graph* (or the *competition-common enemy graph* or the *CCE graph*) of a digraph D is the graph which has the same vertex set as D and has an edge between two distinct vertices x and y if and only if both $N_D^+(x) \cap N_D^+(y) \neq \emptyset$ and $N_D^-(x) \cap N_D^-(y) \neq \emptyset$ hold. See [2,3,7,13] for recent results on double competition graphs.

A *hypergraph $\mathcal{H}$* is a pair $(V(\mathcal{H}), E(\mathcal{H}))$ of a set $V(\mathcal{H})$ of *vertices* and a set $E(\mathcal{H})$ of nonempty subsets of $V(\mathcal{H})$, called *hyperedges*. A hyperedge of the form $\{v\}$ is called a *loop*. We assume that all hypergraphs in this paper have no loops. So all the hyperedges of a hypergraph contain at least two vertices. The notion of competition hypergraphs was introduced by M. Sonntag and H.-M. Teichert [9] in 2004 as another variant of the notion of competition graphs. The *competition hypergraph* of a digraph D is the hypergraph which has the same vertex set as D and in which $e \subseteq V(D)$ is a hyperedge if and only if $|e| \geq 2$ and there exists a vertex v of D such that $e = N_D^-(v)$. See [5,6,10–12] for recent results on competition hypergraphs.

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In this paper, we introduce the notion of the double competition hypergraph of a digraph, and we give characterizations of the double competition hypergraphs of arbitrary digraphs, loopless digraphs, reflexive digraphs, and acyclic digraphs in terms of hyperedge labelings of the hypergraphs.

2. Main results

We define the double competition hypergraph of a digraph as follows.

Definition 1. Let D be a digraph. The *double competition hypergraph* of D is the hypergraph which has the same vertex set as D and in which $e \subseteq V(D)$ is a hyperedge if and only if $|e| \geq 2$ and there exist vertices u and v of D such that $e = N_D^+(u) \cap N_D^-(v)$. $\square$

For a positive integer n, let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

Theorem 1. *Let $\mathcal{H}$ be a hypergraph with n vertices. Then, $\mathcal{H}$ is the double competition hypergraph of an arbitrary digraph if and only if there exist an ordering $(v_1, \ldots, v_n)$ of the vertices of $\mathcal{H}$ and an injective labeling $L : E(\mathcal{H}) \to [n] \times [n]$ of the hyperedge set of $\mathcal{H}$ such that the following condition holds:*

(⋆) for any $i, j \in [n]$, if $|X_i \cap Y_j| \geq 2$, then $X_i \cap Y_j = e_{ij}$,

where e_{ij} denotes the hyperedge e such that $L(e) = (i, j)$ if such e exists, and $e_{ij} = \emptyset$ otherwise, and X_i and Y_j are the sets defined by

$$X_i := \left(\bigcup_{p \in [n]} e_{ip} \right) \cup \{v_b \mid v_i \in e_{ab} \ (a, b \in [n])\}, \tag{1}$$

$$Y_j := \left(\bigcup_{q \in [n]} e_{qj} \right) \cup \{v_a \mid v_j \in e_{ab} \ (a, b \in [n])\}. \tag{2}$$

Proof. First, we show the only-if part. Let $\mathcal{H}$ be the double competition hypergraph of an arbitrary digraph D. Let $(v_1, \ldots, v_n)$ be an ordering of the vertices of D. For $i, j \in [n]$, we define

$$e_{ij} := N_D^+(v_i) \cap N_D^-(v_j). \tag{3}$$

Then e_{ij} is a hyperedge of $\mathcal{H}$ if $|e_{ij}| \geq 2$. Let E^* be the family of e_{ij}'s whose sizes are at least two, i.e.,

$$E^* := \{e_{ij} \mid i, j \in [n], |e_{ij}| \geq 2\}. \tag{4}$$

By the definition of a double competition hypergraph, E^* is the hyperedge set of $\mathcal{H}$. Let $L : E(\mathcal{H}) \to [n] \times [n]$ be the map defined by $L(e_{ij}) = (i, j)$. Then L is injective.

We show that condition (⋆) holds. Fix i and j in $[n]$ and let X_i and Y_j be sets as defined in (1) and (2). Let

$$V_{i*} := \bigcup_{p \in [n]} e_{ip}, \qquad W_i^+ := \{v_b \mid v_i \in e_{ab} \ (a, b \in [n])\},$$

$$V_{*j} := \bigcup_{q \in [n]} e_{qj}, \qquad W_j^- := \{v_a \mid v_j \in e_{ab} \ (a, b \in [n])\},$$

for convenience. Then $X_i = V_{i*} \cup W_i^+$ and $Y_j = V_{*j} \cup W_j^-$. Since $e_{ij} \subseteq X_i$ and $e_{ij} \subseteq Y_j$, it holds that $e_{ij} \subseteq X_i \cap Y_j$. Now we assume that $|X_i \cap Y_j| \geq 2$ and take any vertex $v_k \in X_i \cap Y_j$. There are four cases for v_k arising from the definitions of X_i and Y_j as follows: (i) $v_k \in V_{i*} \cap V_{*j}$; (ii) $v_k \in V_{i*} \cap W_j^-$; (iii) $v_k \in W_i^+ \cap V_{*j}$; (iv) $v_k \in W_i^+ \cap W_j^-$. To show $X_i \cap Y_j \subseteq e_{ij}$, we will check that $v_k \in e_{ij}$ for each case. Consider the case (i). Since $v_k \in V_{i*}$, there exists $p \in [n]$ such that $v_k \in e_{ip}$. Since $v_k \in V_{*j}$, there exists $q \in [n]$ such that $v_k \in e_{qj}$. By (3), we have $v_k \in e_{ip} \cap e_{qj} = N_D^+(v_i) \cap N_D^-(v_p) \cap N_D^+(v_q) \cap N_D^-(v_j) \subseteq N_D^+(v_i) \cap N_D^-(v_j) = e_{ij}$. Consider the case (ii). Since $v_k \in V_{i*}$, there exists $p \in [n]$ such that $v_k \in e_{ip}$. Since $v_k \in W_j^-$, there exists $b \in [n]$ such that $v_j \in e_{kb}$. By (3), we have $v_k \in e_{ip} = N_D^+(v_i) \cap N_D^-(v_p) \subseteq N_D^+(v_i)$ and $v_j \in e_{kb} = N_D^+(v_k) \cap N_D^-(v_b) \subseteq N_D^+(v_k)$, i.e., $v_k \in N_D^-(v_j)$. Therefore $v_k \in N_D^+(v_i) \cap N_D^-(v_j) = e_{ij}$. Consider the case (iii). Since $v_k \in W_i^+$, there exists $a \in [n]$ such that $v_i \in e_{ak}$. Since $v_k \in V_{*j}$, there exists $q \in [n]$ such that $v_k \in e_{qj}$. By (3), we have $v_i \in e_{ak} = N_D^+(v_a) \cap N_D^-(v_k) \subseteq N_D^-(v_k)$, i.e., $v_k \in N_D^+(v_i)$, and $v_k \in e_{qj} = N_D^+(v_q) \cap N_D^-(v_j) \subseteq N_D^-(v_j)$. Therefore $v_k \in N_D^+(v_i) \cap N_D^-(v_j) = e_{ij}$. Consider the case (iv). Since $v_k \in W_i^+$, there exists $a \in [n]$ such that $v_i \in e_{ak}$. Since $v_k \in W_j^-$, there exists $b \in [n]$ such that $v_j \in e_{kb}$. By (3), we have $v_i \in e_{ak} = N_D^+(v_a) \cap N_D^-(v_k) \subseteq N_D^-(v_k)$, i.e., $v_k \in N_D^+(v_i)$, and $v_j \in e_{kb} = N_D^+(v_k) \cap N_D^-(v_b) \subseteq N_D^+(v_k)$, i.e., $v_k \in N_D^-(v_j)$. Therefore $v_k \in N_D^+(v_i) \cap N_D^-(v_j) = e_{ij}$. Thus we obtain $X_i \cap Y_j \subseteq e_{ij}$, and so $X_i \cap Y_j = e_{ij}$. Hence condition (⋆) holds.