Discrete Applied Mathematics 195 (2015) 114–128

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On the generality of the greedy algorithm for solving matroid base problems

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ARTICLE INFO

Article history:
Received 1 October 2013
Received in revised form 7 April 2014
Accepted 27 August 2014
Available online 26 September 2014

Dedicated to our friend and colleague Francesco Maffioli, 1941–2012

Keywords:
Combinatorial optimization
Matroids
Universal objective function
Greedy algorithm
Uniform matroids
Minors
Transversal matroids
Bipartite matchings
Shortest paths

ABSTRACT

It is well known that the greedy algorithm solves matroid base problems for all linear cost functions and is, in fact, correct if and only if the underlying combinatorial structure of the problem is a matroid. Moreover, the algorithm can be applied to problems with sum, bottleneck, algebraic sum or k-sum objective functions.

In this paper, we address matroid base problems with a more general – "universal" – objective function which contains the previous ones as special cases. This universal objective function is of the sum type and associates multiplicative weights with the ordered cost coefficients of the elements of matroid bases such that, by choosing appropriate weights, many different – classical and new – objectives can be modeled. We show that the greedy algorithm is applicable to a larger class of objective functions than commonly known and, as such, it solves universal matroid base problems with non-negative or non-positive weight coefficients. Based on problems with mixed weights and a single $(-,+)$-sign change in the universal weight vector, we give a characterization of uniform matroids. In case of multiple sign changes, we use partition matroids. For non-uniform matroids, single sign change problems can be reduced to problems in minors obtained by deletion and contraction. Finally, we discuss how special instances of universal bipartite matching and shortest path problems can be tackled by applying greedy algorithms to associated transversal matroids.

1. Introduction

Quoting from the preface of Welsh [27], matroids play a "unifying and central role … in combinatorial theory" as various problems in combinatorial optimization can be modeled as matroid base problems and, therefore, be solved efficiently by greedy algorithms. For a matroid $M = (E, \mathcal{B})$ given by its ground set E and its collection of bases $\mathcal{B}$, the classical problem is the *minimum matroid base problem (MMBP)*

$$\min_{B\in\mathcal{B}} \sum_{e\in B} c(e), \tag{1}$$

where costs $c(e) \in \mathbb{R}$ are assigned to the elements $e \in E$. A base $B^* \in \mathcal{B}$, which is optimal to (1), is called a *minimum-cost base*.

The goal of this paper is to show that the greedy algorithm – which starts with the empty set and iteratively adds an element of smallest possible cost while preserving independence of the set – is not only correct for matroid base problems

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http://dx.doi.org/10.1016/j.dam.2014.08.034

with sum objective function, but also for many other objectives. Among them are, for instance, bottleneck, algebraic sum, k-sum, k-max, cent-dian and trimmed-mean objective functions.

To this end, the remainder of the paper is organized as follows.

In Section 2, we introduce a unified framework to formulate matroid base problems with different types of objective functions. Based on ordered weighted averaging operators and ordered median functions, these universal matroid base problems minimize sum objectives, where multiplicative weights are associated with the ordered cost coefficients of the elements of matroid bases. The solvability by greedy methods will heavily depend on the chosen weight coefficients.

In Section 3, we review the (standard) greedy algorithm. Its power is illustrated in Section 4, where universal matroid base problems with either non-negative or non-positive weight coefficients are solved. As opposed to this, we give an example to demonstrate that the "pure" greedy strategy fails if there are strictly positive and strictly negative weights. Consequently, the subsequent sections focus on problems with mixed weights.

We distinguish between single (Section 5) and multiple (Section 6) sign changes in the universal weight vectors as well as between uniform and non-uniform matroids. For uniform matroids, we prove the result that a matroid base composed by a minimum-cost and maximum-cost base is optimal for problems with one $(-,+)$-sign change. We also show that this is only true for uniform matroids, thus providing a new characterization of this matroid class. These results can be carried over to problems with multiple sign changes, however, a solution in appropriately defined partition matroids becomes necessary. For non-uniform matroids, we use deletion and contraction operations if the universal weight vector changes its sign only once.

In Section 7, we consider two applications of transversal matroids, the universal bipartite matching problem and the universal shortest path problem. Our results are summarized in Section 8.

For notations as well as basic definitions and results from matroid theory we refer to the book of Oxley [19]. The paper is based on the thesis of Turner [23].

2. Universal matroid bases

Given a matroid $M = (E, \mathcal{B})$, the focus is on matroid base problems with universal objective function generalizing the well-known sum objective function and including several more as special cases.

The definition is based on two parts, the sorting of the cost coefficients of the elements of each base $B \in \mathcal{B}$ and their multiplication with universal weights $\lambda_1, \ldots, \lambda_r$. The rank of matroid $M = (E, \mathcal{B})$, which is equal to the cardinality of its bases, is denoted by r.

Definition 1. Given costs $c(e) \in \mathbb{R}$ for all elements $e \in E$ and a base $B \in \mathcal{B}$, the *sorted cost vector* (with respect to $c(e), e \in E$, and B) is

$$c_{\geq}(B) := (c_{(1)}(B), \ldots, c_{(r)}(B))$$

where $c_{(i)}(B), i = 1, \ldots, r$, is the ith largest cost coefficient of base B.

Combining the sorted costs with a given set of weights, we get a universal objective function.

Definition 2. Given a matroid $M = (E, \mathcal{B})$ of rank r with

- costs $c(e) \in \mathbb{R}$ for all $e \in E$ and
- weights $\lambda_i \in \mathbb{R}$ for all $i = 1, \ldots, r$,

the *universal minimum matroid base problem (Univ-MMBP)* is

$$\min_{B \in \mathcal{B}} f_\lambda(B) := \sum_{i=1}^{r} \lambda_i \cdot c_{(i)}(B). \tag{2}$$

An optimal base $B^* \in \mathcal{B}$ is called a *universal minimum-cost base*.

Observe that, for a base $B := \{b_1, \ldots, b_r\} \in \mathcal{B}$ with $c(b_1) \leq \cdots \leq c(b_r)$ or $c(b_1) \geq \cdots \geq c(b_r)$, the objective function in (2) can be reformulated as

$$f_\lambda(B) := \sum_{i=1}^{r} \lambda_i \cdot c(b_{r-i+1})$$

or

$$f_\lambda(B) := \sum_{i=1}^{r} \lambda_i \cdot c(b_i),$$

respectively.