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Weighted Coxeter graphs and generalized geometric representations of Coxeter groups

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ABSTRACT

We introduce the notion of weighted Coxeter graph and associate to it a certain generalization of the standard geometric representation of a Coxeter group. We prove sufficient conditions for faithfulness and non-faithfulness of such a representation. In the case when the weighted Coxeter graph is balanced we discuss how the generalized geometric representation is related to the numbers game played on the Coxeter graph.

1. Introduction and preliminaries

The theory of Coxeter groups is connected with different fields of mathematics: algebra, geometry, combinatorics, graph theory. In this work we study certain linear representations of Coxeter groups and relate them to the corresponding Coxeter graphs.

From the algebraic point of view a Coxeter system (W, S) is a group W with the set of generators $S = \{s_1, s_2, \dots, s_n\}$ and relations $s_i^2 = 1$, $(s_is_j)^{m_{ij}} = 1$, where $2 \leqslant m_{ij} \in \mathbb{N}$ or $m_{ij} = \infty$, the latter means that there is no relation between s_i and s_j.

Geometrically, Coxeter groups can be viewed as discrete groups generated by orthogonal reflections in a vector space with a pseudo-Euclidean metric.

A Coxeter group can also be defined by its Coxeter graph. Its vertex set coincides with the set S of Coxeter generators. Two vertices s_i and s_j are not connected if $m_{ij} = 2$, are connected by an unlabeled edge if $m_{ij} = 3$ and are connected by an edge labeled by m_{ij} if $m_{ij} > 3$. Edges labeled by m_{ij} with $m_{ij} > 3$ are called *multiple edges*. A Coxeter group is called *simply laced* if its Coxeter graph does not have multiple edges, i.e., m_{ij} equals to 2 or 3 for all $i \neq j$.

A combinatorial model of the Coxeter group can be given by the so called Mozes' *numbers game* which we briefly describe now. Consider a graph with vertices indexed by $1, 2, \dots, n$ and numbers x_j associated to its vertices. The column vectors

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$(x_1, x_2, \ldots, x_n)^t$ are called configurations or game positions. The (legal) "moves" in the game (which are also called "firings") are local rearrangements of the assigned values at a chosen node and its neighbors: the move which corresponds to a vertex j consists of adding the value x_j multiplied by a certain positive weight k_{ij} to each value x_i, i.e., $x_i := x_i + k_{ij}x_j$, where the vertices j and i are neighbors, and then reversing the sign, i.e., $x_j := -x_j$. Using Weyl groups and Kac–Moody algebras, see [4,9] for definitions and other details, Mozes in [12] gave an algebraic characterization of the initial positions giving rise to finite games and proved that for those the number of steps and the final configuration do not depend on the moves of the player. It was developed as a generalization of the following problem of the International Mathematical Olympiad (IMO) in 1986: five integers with positive sum are assigned to the vertices of a pentagon. If there is at least one negative number, the player may pick one of them, say y, add it to its two neighbors x and z, and then reverse the sign of y. The game terminates when all the numbers are nonnegative. Prove that this game must always terminate. Several solutions to this problem can be found in [14]. Eriksson and Björner found deep connections between the numbers game and Coxeter groups: taking the positive weights k_{ij} satisfying $k_{ij}k_{ji} = 4\cos^2\left(\pi/m_{ij}\right)$ where m_{ij} is the minimal number such that for Coxeter generators s_i and s_j there is the relation $\left(s_i s_j\right)^{m_{ij}} = 1$ and $k_{ij}k_{ji} \geqslant 4$ when the element $s_i s_j$ has infinite order, the numbers game becomes a combinatorial model of the Coxeter group, where group elements correspond to positions and reduced decompositions correspond to legal play sequences, see Chapter 4 in [3,7] for details.

Any Coxeter group has the canonical linear representation called the *geometric representation* defined in the following way. The dimension of this representation equals to the number of Coxeter generators. For each pair of generators (s_i, s_j), which do not commute, choose a *positive* number k_{ij} such that $k_{ij}k_{ji} = 4\cos^2\left(\pi/m_{ij}\right)$; if $m_{ij} = \infty$, put $k_{ij} = k_{ji} = 2$. Each generator s_i is mapped to the matrix σ_i which differs from the identity matrix only in its ith row. The diagonal element in the position (i, i) is -1, and for $i \neq j$ the entry in the position (i, j) equals to k_{ij}; if the generators s_i and s_j commute, then the entry in the position (i, j) is zero. Numbers k_{ij} and k_{ji} may be different. If all the numbers m_{ij} are 2, 3, 4, 6 or ∞ it is possible to choose all k_{ij} integers. We emphasize that the numbers k_{ij} are positive because it is crucial for the following theorem which is well known (see [3]):

Theorem. *Matrices σ_i satisfy the Coxeter relations, i.e., the mapping $s_i \mapsto \sigma_i$ is a representation. The representation $s_i \mapsto \sigma_i$ is faithful. The representation $s_i \mapsto \sigma_i$ is called the* geometric representation.

If for any i, j, $k_{ij} = k_{ji} = 2\cos\left(\pi/m_{ij}\right)$, then the representation $s_i \mapsto \sigma_i$ is called the standard geometric representation. When the group is simply laced, all non-zero entries of matrices σ_i of the standard geometric representation are ± 1.

The connection between the numbers game and the standard geometric representation is as follows: it is easy to see that a move associated with the vertex i in the described above numbers game is the action of the matrix σ_i^t on the column vector of configuration $(x_1, x_2, \ldots, x_n)^t$.

This paper is devoted to a natural generalization of the objects defined above, obtained by allowing the numbers k_{ij} which are used in the definition of the geometric representation and numbers game to take not necessarily real positive values. We define a corresponding version of the numbers game and a generalization of the standard geometric representation and give sufficient conditions for its faithfulness and non-faithfulness. In the next section we give main definitions and formulate the results. We freely use the standard terminology and results on Coxeter groups, Coxeter graphs etc. applied in the monographs [4,9,8,3].

2. Weighted Coxeter graph and the corresponding representation

In this paper we shall consider geometric representations and number games associated to *weighted Coxeter graphs*. Let us associate weights to the edges of Coxeter graphs as follows.

Denote by $\bar{E}$ the set of directed edges of the graph $\Gamma = (V, E)$, i.e.,

$$\left\{s_i, s_j\right\} \in E \iff \left(s_i, s_j\right), \quad \left(s_j, s_i\right) \in \bar{E}.$$

Definition 2.1. Let $\Gamma = (V, E)$ be the Coxeter graph of the Coxeter system (W, S). Define a *legal weight function* to be a function

$$f : \bar{E} \to \mathbb{C} \setminus \{0\} \quad \text{such that}\; f\left(\left(s_i, s_j\right)\right) = \left(f\left(\left(s_j, s_i\right)\right)\right)^{-1}.$$

Notice that the complex numbers $\left\{f\left(\left(s_i, s_j\right)\right)\right\}$ and the integers $\left\{m_{ij}\right\}$ (when $m_{ij} > 3$) are two independent sets of weights on the edges of our graph.

Definition 2.2. We say that the triple $\Gamma_f = (V, E, f)$ is a *weighted Coxeter graph* if $\Gamma = (V, E)$ is a Coxeter graph and f is a legal weight function on its edges.

Definition 2.3. For any path in a weighted Coxeter graph we can define the *weight of the path* as the product of all weights of the edges of which this path consists. We call a weighted Coxeter graph *balanced* if the weight of any closed path in this graph is equal to 1.