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Maximum probability shortest path problem☆

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The maximum probability shortest path problem involves the constrained shortest path problem in a given graph where the arcs resources are independent normally distributed random variables. We maximize the probability that all resource constraints are jointly satisfied while the path cost does not exceed a given threshold. We use a second-order cone programming approximation for solving the continuous relaxation problem. In order to solve this stochastic combinatorial problem, a branch-and-bound algorithm is proposed, and numerical examples on randomly generated instances are given.

1. Introduction

Shortest path problem (SPP for short) is a fundamental problem in combinatorial optimization. Though interesting in its own right, algorithms for this problem are also used as building blocks in the design of algorithms for a large number of industrial complex problems. As a result, there has been an extensive literature on SPP and its various aspects [6,9,10]. SPP is solvable in polynomial time. However, it becomes NP-hard when one or more additional constraints are added [13]. In this paper, we focus on the resource constrained shortest path problem (RCSP hereafter), especially when a subset of resource constraints parameters is random variables. The deterministic RCSP has attracted considerable attention in the literature [3,14,16,21,22,28,34]. RCSP is NP-complete even for the case of one resource [13].

Handler and Zang [13] gave an exact algorithm based upon a Lagrangian relaxation, while Hassin [15] presented a pseudo-polynomial algorithm. Jensen and Berry [19] proposed an algorithm which combined dynamic programming and the use of dominance approach. Aneja et al. [2] presented a generalization of the Dijkstra algorithm to solve the problem. Beasley and Christofides [5] presented a branch-and-bound approach based on a Lagrangian relaxation. Avella et al. [3] proposed a heuristic algorithm to approximate the problem, which is based on the extension to the discrete case of an exponential penalty function.

In the deterministic SPP (or RCSP), all the parameters (distance, time or cost) are known. However in real world applications, the parameters are not known in advance due to different uncertainty factors, e.g., travel times between two cities. Therefore, it is natural to consider these parameters as random variables, which turn the underlying problem into a stochastic optimization problem [31]. Stochastic shortest path problem (SSPP) has been widely studied in the literature. Hutsona and Shierb [18], Mirchandani et al. [23] and Murthy et al. [24] considered the problem of selecting a path which maximizes utility functions or minimizes cost functions. Ohtsubo [26,27] selected a probability distribution over the set of successor nodes and formulated this problem as a Markov decision process. Provan [30], Polychronopoulos and Tsitsiklis [29]

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studied the expected shortest paths in the networks, where information on arc cost values is accumulated as the graph is traversed.

In this paper, we study a variant of stochastic resource constrained shortest path problem (called SRCSP for short), namely the maximum probability of resource constraints. We are given a simple and acyclic digraph $G = (V, A)$, a source node s, a sink node t, a threshold of the cost function C, and K resource limits d_1 to d_K. Each arc e has a deterministic cost $c(e)$ and resource consuming $\tilde{a}_k(e)$ units of resource k where $1 \le k \le K$. We assume that the resources consumed by traversing arcs are independently normally distributed, i.e., $\tilde{a}_j(e_1)$ and $\tilde{a}_i(e_2)$ are independent provided $i \neq j$. The objective is to maximize the probability of all resource constraints to be jointly satisfied while the path cost does not exceed a given threshold. As the deterministic RCSP is NP-hard [13], SRCSP is also NP-hard by choosing all the resource parameter means of the arcs equal to zero. To the best of our knowledge, there is only one paper where the probability that a resource constraint is satisfied, is maximized using a quasi-convex method [25].

We propose a branch and bound framework to solve the maximum probability RCSP. In branch-and-bound method, bounding plays a crucial role. So special interest is given to its relaxation problems. As the linear relaxation of this problem (where $x \in \{0, 1\}^{|A|}$ is replaced by $x \in [0, 1]^{|A|}$) is generally not convex, we propose an efficient convex approximation to come up with tight upper bounds. Moreover, we improve the convexity results of Henrion and Strugarek [17]. Furthermore, numerical experiments on a set of generated instances from the literature are conducted to illustrate the efficiency of our approach. The remainder of the paper is organized as follows. The mathematical formulation of SRCSP is given in Section 2. In Sections 3 and 4, the convex relaxations of SRCSP are presented for the case of individual chance constraint, i.e., $K = 1$ and for the case of joint chance constraints, i.e., $K > 1$ respectively. In Section 5, the standard approximation of the individual constraint maximum probability is introduced together with solving method. In Section 6, numerical results are given to compare the two relaxations either with individual or joint chance constraints. The conclusions are given in the last section.

2. SRCSP formulation

SRCSP can be formulated as a stochastic combinatorial optimization problem in the following way: let $x \in \{0, 1\}^{|A|}$ such that each component x_a of x represents an arc $a \in A$. For a directed path P, we define the corresponding $x = x(P)$ such that $x_a = 1$ if and only if $a \in P$. Then, SRCSP can be mathematically formulated as follows:

$$\begin{aligned} \max\quad & \boldsymbol{Pr}\{\tilde{a}_k^T x \le d_k, k = 1, \dots, K\} \\ & c^T x \le C && (1) \\ \text{s.t.}\quad & Mx = b && (2) \\ & x \in \{0, 1\}^n && (3) \end{aligned}$$

where $c \in R^n$, $C \in R$, $M \in \mathbb{R}^{m\times n}$ is the *node–arc incidence matrix* [1], and $b \in \mathbb{R}^m$ is a vector where all elements are 0 except the s-th and the t-th components which are equal to 1 and -1 respectively. $\tilde{a}_k \in R^n$, $k = 1, \dots, K$ are independent random vectors in R^n with a multivariate normal distribution with a known mean μ_k and a known covariance matrix V_k; $d_k \in R$, $k = 1, \dots, K$.

This problem can be reformulated as:

$$\begin{aligned} \max\quad & p \\ \text{s.t.}\quad & p \le \boldsymbol{Pr}\{\tilde{a}_k^T x \le d_k,\ k = 1, \dots, K\} && (4) \\ \text{SRCSP}\quad & c^T x \le C && (5) \\ & Mx = b \\ & x \in \{0, 1\}^n. \end{aligned}$$

Assumptions. Let x^* be the optimal solution of SRCSP, we assume $\frac{1}{2} \le \boldsymbol{Pr}\{\tilde{a}_k^T x^* \le d_k\}$, for any $k \in \{1, \dots, K\}$. In other words, the resource threshold is at least as large as the expectation of the resources consumed by traversing the path.

We consider two different variants of SRCSP: the case where $K = 1$, i.e., individual probabilistic constraint; and the case of joint chance constraints where $K > 1$.

3. SRCSP with individual chance constraint

In this section, we consider the individually probabilistic SRCSP, i.e., $K = 1$. Before giving the mathematical formulation of the problem, we introduce a small instance to illustrate SRCSP problem as in Fig. 1. In each arc, there is not only one deterministic cost but also a random resource consumption which is assumed to be normally distributed. For instance, for arc (1, 2), the cost is 1 and the resource consumption is normally distributed with mean 3 and variance 1. For the sake of simplicity, all the resource consumptions of the arcs are assumed to be independent. The threshold of path costs C is set to