Discrete Applied Mathematics 192 (2015) 77–81

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Characterization of minimum cycle basis in weighted partial 2-trees[☆,☆☆]

N.S. Narayanaswamy, G. Ramakrishna[*]

Department of Computer Science and Engineering, Indian Institute of Technology Madras, India

ARTICLE INFO

Article history:
Received 21 February 2013
Received in revised form 23 March 2014
Accepted 1 May 2014
Available online 16 May 2014

Keywords:
Minimum cycle basis
Partial 2-trees
Lex short cycles

ABSTRACT

For a weighted outerplanar graph, the set of lex short cycles is known to be a minimum cycle basis (Liu and Lu, 2010). In our work, we show that the set of lex short cycles is a minimum cycle basis in weighted partial 2-trees (graphs of treewidth at most two), which is a superclass of outerplanar graphs. In general graphs, a minimum cycle basis is a subset of the set of lex short cycles, where as the equality is known to hold only for the weighted outerplanar graphs.

1. Introduction

A cycle basis is a compact description of the set of all cycles of a graph and has various applications including the analysis of electrical networks [7]. Let $G = (V(G), E(G))$ be an edge weighted graph and let $m = |E(G)|$ and $n = |V(G)|$. A *cycle* is a connected graph in which the degree of every vertex is two. Let $E(G) = \{e_1, \ldots, e_m\}$. An *incidence vector* $(x_1, \ldots, x_m)$ is associated with every cycle C in G, where for each $1 \le i \le m$, x_i is 1 if $e_i \in E(C)$ and 0 otherwise. The *cycle space* of G is the vector space over $\mathbb{F}_2^m$ spanned by the incidence vectors of cycles in G. A *cycle basis* of G is a minimum set of cycles whose incidence vectors span the cycle space of G. The weight of a cycle C is the sum of the weights of the edges in C. A cycle basis $\mathcal{B}$ of G is a *minimum cycle basis* (MCB) if the sum of the weights of the cycles in $\mathcal{B}$ is minimum.

Motivation. For a weighted graph G, Horton has identified a set $\mathcal{H}$ of $O(mn)$ cycles and has shown that a minimum cycle basis of G is a subset of $\mathcal{H}$ [6]. Liu and Lu have shown that the set of *lex short cycles* (defined later) is a minimum cycle basis in weighted outerplanar graphs [9]. We generalize this result for partial 2-trees, which is a superclass of outerplanar graphs.

Our contribution. The following are the main results in this work.

Theorem 1.1. *Let G be a weighted partial 2-tree on n vertices and m edges. Then the number of lex short cycles in G is $m - n + 1$.*

Theorem 1.2. *For a weighted partial 2-tree G, the set of lex short cycles is a minimum cycle basis.*

Related work. The characterization of graphs using cycle basis was initiated by MacLane [10]. In particular, MacLane showed that a graph G is planar if and only if G contains a cycle basis B, such that each edge in G appears in at most two cycles of B.

☆ A preliminary version of this paper appeared as *Characterization of Minimum Cycle Basis in Weighted Partial 2-trees* in the proceedings of CTW 2012.

☆☆ Supported by the Indo-Max Planck Centre for Computer Science Programme in the area of *Algebraic and Parameterized Complexity* for the year 2012–2013.

* Corresponding author.
E-mail addresses: swamy@cse.iitm.ac.in (N.S. Narayanaswamy), rama.vikram@gmail.com, grama@cse.iitm.ac.in (G. Ramakrishna).

http://dx.doi.org/10.1016/j.dam.2014.05.008
0166-218X/

However, he referred to a cycle basis as a *complete independent set of cycles*. Formally, the concept of cycle space in graphs was introduced in [3] after four decades. Later, it was characterized that a planar 3-connected graph G is a Halin graph if and only if G has a planar basis B, such that each cycle in B has an external edge [13]. There after, it was shown that every 2-connected outerplanar graph has a unique MCB [8]. Subsequently, it was proven that Halin graphs that are not necklaces have a unique MCB [12].

The first polynomial time algorithm for finding an MCB was given by Horton [6]. Since then, many improvements have taken place on algorithms related to minimum cycle basis and its variants. A detailed survey of various algorithms, characterizations and the complexity status of cycle basis and its variants was compiled by Kavitha et al. [7]. The current best algorithm for MCB runs in $O(m^2n/\log n)$ time and is due to Amaldi et al. [1].

Graph preliminaries. In this paper, we consider only simple, finite, connected, undirected and weighted graphs. We refer [14] for standard graph theoretic terminologies. Let G be an edge weighted graph. Let $X \subseteq V(G)$ be a set of vertices. $G-X$ denotes the graph obtained after deleting the set of vertices in X from G. We use $G[X]$ to denote the subgraph induced by the vertices in X. A set $X \subseteq V(G)$ of vertices is a *vertex separator* if $G - X$ is disconnected. A *path* is a connected graph that has two vertices of degree one and the rest being two. We use $P(u, v)$ to denote a path joining the vertices u and v. If the end vertices of the path under consideration are clear from the context, then we simply use P to denote such a path. A *component* of G is a maximal connected subgraph. K_3 denotes a cycle on 3 vertices and K_2 denotes an edge. $K_{2,3}$ is a complete bipartite graph (V_1, V_2) such that $|V_1| = 2$, $|V_2| = 3$. A graph is *planar* if it can be drawn on the plane without any edge crossings. A planar graph is *outerplanar* if it can be drawn on the plane such that all of its vertices lie on the boundary of its exterior region. A 2-tree is defined inductively as follows: K_3 is a 2-tree; if G' is a 2-tree and $(x, y) \in E(G')$, then the graph with the vertex set $V(G') \cup \{u\}$ and edge set $E(G') \cup \{(u, x), (u, y)\}$ is a 2-tree. A graph is a *partial 2-tree* if it is a subgraph of a 2-tree. Alternatively, a graph of treewidth (defined in [11]) at most two is a *partial 2-tree*. An *H-subdivision* (or subdivision of H) is a graph obtained from a graph H by replacing edges with pairwise internally vertex disjoint paths.

2. On lex short cycles in weighted partial 2-trees

We first define lex shortest paths and lex short cycles and explore their structural properties in weighted partial 2-trees. Then by using these structural properties, a procedure is defined to decompose a weighted partial 2-tree. This procedure helps in computing the number of lex short cycles in weighted partial 2-trees and further proves the main result.

Let G be a weighted graph associated with a weight function $w : E(G) \to \mathbb{N}$. For a totally ordered set S, $\min(S)$ denotes the minimum element in S. Let $V(G)$ be a totally ordered set. The notion of lex shortest path and lex short cycle presented here is from [5]. A path $P(u, v)$ between two distinct vertices u and v is *lex shortest path* if for all the paths P' between u and v other than P, exactly one of the following three conditions hold: (1) $w(P') > w(P)$, (2) $w(P') = w(P)$ and $|E(P')| > |E(P)|$, and (3) $w(P') = w(P)$, $|E(P')| = |E(P)|$ and $\min(V(P') \setminus V(P)) > \min(V(P) \setminus V(P'))$, where $w(P) = \Sigma_{e \in E(P)} w(e)$. The lex shortest path between any two vertices u and v is unique. For a subgraph H of G, $lsp_H(x, y)$ denotes the lex shortest path between vertices x and y in H; $lsp(u, v)$ denotes the lex shortest path between x and y in G. A cycle C is *lex short* if for every two vertices u and v in C, $lsp(u, v)$ is in C. The set of lex short cycles of G is denoted by $LSC(G)$. For a subgraph H of G, the total order of $V(H)$ is the order induced by the total order of $V(G)$. For any connected graph, the cardinality of a cycle basis is $m - n + 1$. Thus by the following two lemmas, the set of lex short cycles in outerplanar graphs is a minimum cycle basis.

Lemma 2.1 ([5]). *A minimum cycle basis of a weighted graph G is a subset of $LSC(G)$.*

Lemma 2.2 ([9]). *For a simple weighted outerplanar graph G, $|LSC(G)| = m - n + 1$.*

We now present our lemmas and theorems that are required to prove the main results.

Lemma 2.3. *Let G be a graph and $\{u, v\}$ be a vertex separator in G. Let P be a path between u and v. Then there exist one component H in $G - \{u, v\}$, such that $V(P) \cap V(H) = \emptyset$ and $E(P) \cap E(H) = \emptyset$.*

Proof. Since $\{u, v\}$ is a vertex separator, the number of components in $G - \{u, v\}$ is at least two. If there are no internal vertices in P, then none of the components in $G - \{u, v\}$ contain $V(P)$ and $E(P)$. Consider the other case where P has at least one internal vertex. Since the end vertices of P are u and v, and P is a path, the graph $P - \{u, v\}$ is a path in $G - \{u, v\}$. Since $\{u, v\}$ is a vertex separator and $P - \{u, v\}$ is a path, it follows that $P - \{u, v\}$ is in a single component in $G - \{u, v\}$. Then we have at least one component H in $G - \{u, v\}$, such that $V(P) \cap V(H) = \emptyset$ and $E(P) \cap E(H) = \emptyset$. □

Lemma 2.4. *Let G be a partial 2-tree that is not outerplanar. Then there exists a $K_{2,3}(\{u, v\}, \{x, y, z\})$-subdivision in G, such that $G - \{u, v\}$ contains at least three components.*

Proof. A graph is outerplanar if and only if it contains no subgraph that is a subdivision of K_4 or $K_{2,3}$ [2]. Since a partial 2-tree does not contain a subdivision of K_4, a partial 2-tree is outerplanar if and only if it does not contain a subdivision of $K_{2,3}$. Consider a $K_{2,3}(\{u, v\}, \{x, y, z\})$-subdivision H in G. Since H is a $k_{2,3}$-subdivision, we have three internally vertex disjoint paths P_x, P_y and P_z between u and v that contains x, y and z respectively. Assume that $G-\{u, v\}$ has at most two components. In $G-\{u, v\}$, without loss of generality, let x and y be in a single connected component. Let (x', y') be a closest pair of vertices (with respect to the distance between them) such that $x' \in V(P_x)$ and $y' \in V(P_y)$. Let $P(x', y')$ be a shortest path between x' and y'.