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Hamiltonian cycles in unitary prefix transposition rearrangement graphs☆

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ABSTRACT

Cayley graphs have been extensively studied by graph and group theorists, computer scientists, molecular biologists and coding theorists. We focus on two challenging problems on Cayley graphs arising on sequence comparison: hamiltonian cycle and graph diameter. A unitary prefix transposition exchanges two adjacent blocks in a permutation such that one block contains the first elements and one of the blocks is unitary. The Unitary Prefix Transposition Rearrangement Graph has the permutations in the Symmetric Group S_n as its vertex set and two vertices are adjacent if there exists a unitary prefix transposition that applied to a permutation produces the other one. We show that this Cayley graph has diameter $n-1$ and is hamiltonian.

1. Introduction

Sequence comparison is well-studied in computer science and the goal in the edit distance problem is to find the minimum number of local operations that transform a given sequence into another one. More recently, global operations such as reversals and transpositions have been considered, known as rearrangement events, consisting in comparing the genomes of two species. The rearrangement event known as transposition exchanges two adjacent blocks of any length in a genome. An arbitrary genome formed by n genes is represented by a permutation on n elements. The *sorting by transpositions* problem asks for the minimum number of transpositions to transform a permutation into the identity permutation and is considered a challenging problem in computational biology [2], just recently settled as NP-hard [4]. Consequently, some variants of this problem have been studied, in the hope that they would shed some light on the sorting by transpositions problem [3]. A variation of the problem called *sorting by prefix transpositions* considers only transpositions that move the first consecutive elements of the genome [6]. Sorting by prefix transpositions is still an open problem, for which only some tight bounds for the distance and diameter are known [5,9]. In the present paper, we study the sorting by unitary prefix transpositions problem where the prefix transpositions have a block of length 1.

We call a graph constructed with the unitary prefix transpositions as *Unitary Prefix Transposition Rearrangement Graph*, denoted by $URG(n)$, whose vertices are the permutations in the Symmetric Group S_n and two vertices are adjacent if there is a

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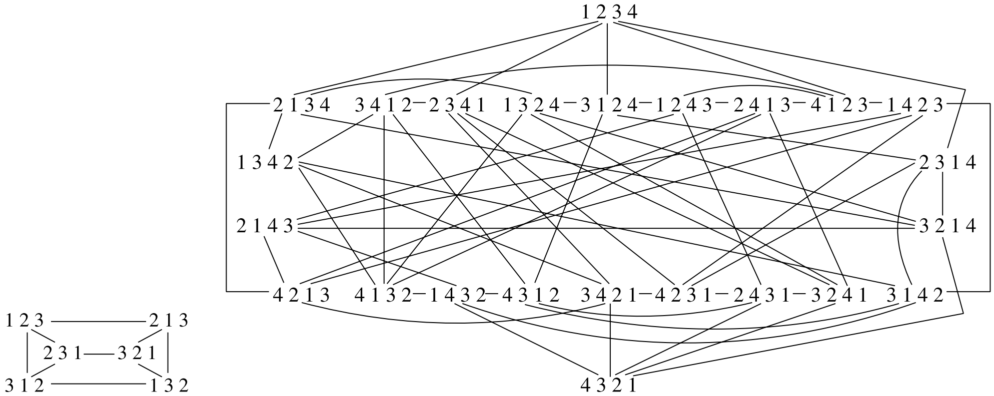

(a) *URG*(3). (b) *URG*(4).

Fig. 1. Unitary Prefix Rearrangement Graphs *URG*(n).

unitary prefix transposition that applied to a permutation produces the other one (Fig. 1). Analogously, we may define other two graphs *TRG*(n) and *PRG*(n) whose vertices are the permutations in the Symmetric Group S_n but whose edges consider all transpositions and all prefix transpositions, respectively. Clearly, *URG*(n) is a spanning subgraph of *PRG*(n), and *PRG*(n) is a spanning subgraph of *TRG*(n).

Cayley graphs are connected vertex-transitive graphs that provide a natural and rich framework for the study of abstract groups which are described by a generating set [10]. In molecular biology, Cayley graphs on the symmetric group S_n have been considered since permutations are used to represent sequences of genes in chromosomes and genomes, and some operations on permutations represent evolutionary events providing the suitable generating set [8]. We prove that the graph *URG*(n) is a Cayley graph and, therefore, is connected vertex-transitive.

A spanning cycle in a graph is called a *hamiltonian cycle* and a graph which contains such a cycle is said to be *hamiltonian*. A *hamiltonian path* is a path that contains every vertex of the graph precisely once. Determining if a graph has a hamiltonian cycle is an NP-Complete problem. A conjecture due to Lovász [11] claims that every connected vertex-transitive graph has a hamiltonian path. Therefore, the study of hamiltonian cycles and paths in Cayley graphs may provide additional evidence to support Lovász conjecture, or offer a counterexample for disproving it. There are only four known connected vertex-transitive graphs which do not have a hamiltonian cycle, and all of these graphs have a hamiltonian path. They are the Petersen graph, the Coxeter graph and the graphs obtained from each of these two graphs by replacing each vertex with a triangle. Since none of these graphs is a Cayley graph, a related problem asks whether every Cayley graph on a finite group is hamiltonian [8,11].

In Section 2, we prove that the Unitary Prefix Transposition Rearrangement Graph *URG*(n) is a Cayley graph and consists of n isomorphic copies of *URG*($n-1$). Also, we show that the diameter of *URG*(n) is $n-1$. In Section 3, we prove the existence of $2n$-cycles that will be extended to the desired hamiltonian cycle in *URG*(n), which gives a hamiltonian cycle in *PRG*(n) as well. Section 4 concludes the paper.

2. The unitary prefix transposition rearrangement graph

A *permutation* is a bijective function of a finite set onto itself. Throughout this paper, we consider permutations of $[n] = \{1, 2, \dots, n\}$. Denote $\pi = [\pi_1 \pi_2 \dots \pi_i \dots \pi_n]$ as a permutation of n elements that maps 1 to π_1, 2 to $\pi_2, \dots, i$ to $\pi_i, \dots, n$ to π_n. Since π is bijective, if $i \neq j$, then $\pi_i \neq \pi_j$. We define the product of two permutations as a composition to the right. Therefore, the product $\pi\sigma$ is the composition of two functions, in which π is applied first, and then σ. The set of all permutations of n elements along with the product operation forms a symmetric group, denoted by S_n.

A *transposition*, denoted by $t(i, j, k)$, for $1 \leqslant i < j < k \leqslant n+1$, is the permutation $t(i, j, k) := [1\ 2 \dots i-1\ j\ j+1 \dots k-1\ i\ i+1 \dots j-1\ k \dots n]$, if $k < n+1$ or $t(i, j, k) := [1\ 2 \dots i-1\ j\ j+1 \dots k-1\ i\ i+1 \dots j-1]$, if $k = n+1$. The transposition $t(i, j, k)$ "cuts" the elements between the positions j and $k-1$ (both inclusive) and "pastes" them immediately before the i-th position. If $\pi = [\pi_1\pi_2 \dots \pi_{i-1}\pi_i \dots \pi_{j-1}\boxed{\pi_j \dots \pi_{k-1}}\dots] \in S_n$, then $\pi \cdot t(i, j, k) = [\pi_1\pi_2 \dots \pi_{i-1}\boxed{\pi_j \dots \pi_{k-1}}\pi_i \dots \pi_{j-1} \dots]$. A *prefix transposition* [5,6], denoted by $pt(j, k)$, is a transposition $t(1, j, k)$, where $1 < j < k \leqslant n+1$. The *Prefix Transposition Rearrangement Graph*, denoted by *PRG*(n), is a graph where vertices are the permutations in S_n and two vertices π and σ are adjacent if there exists a transposition $pt(j, k)$ such that $\sigma = \pi \cdot pt(j, k)$.

A *unitary prefix transposition*, denoted by $ut'(k)$ or $ut''(k)$ is a transposition $pt(j, k)$, where $j = 2$ or $j = k-1$, respectively. Note that $ut'(k) = pt(2, k)$ is the inverse permutation of $ut''(k) = pt(k-1, k)$. The *Unitary Prefix Transposition Rearrangement Graph*, denoted by *URG*(n), is a graph where vertices are the permutations in S_n and two vertices π and σ are adjacent if there exists a unitary prefix transposition $ut'(k)$ or $ut''(k)$ such that $\sigma = \pi \cdot ut'(k)$ or $\sigma = \pi \cdot ut''(k)$ (Fig. 1).