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Asymptotic surviving rate of trees with multiple fire sources

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ABSTRACT

For Hartnell's firefighter game with f vertices initially on fire and at most d defended vertices per round, the surviving rate $\rho(G, f, d)$ of a graph G is the average proportion of its vertices that can be saved in the game on G, where the average is taken over all sets of f fire sources. Cai et al. (2010) showed that $\rho(T, 1, 1) = 1 - O\left(\frac{\log n}{n}\right) = 1 - o(1)$ for every tree T of order n.

We study the maximum value $c(f, d)$ such that $\rho(T, f, d) \geq c(f, d) - o(1)$ for every tree T of order n, where the $o(1)$ term tends to 0 as n tends to infinity. In this notation, the result of Cai et al. states $c(1, 1) = 1$. Our main results are that $c(f, 1) \geq \left(\frac{1}{3}\right)^f$ and that $\frac{4}{9} \leq c(2, 1) \leq \frac{3}{4}$.

1. Introduction

Hartnell's *firefighter game* [9] is a popular one-person discrete contagion containment game. The game has two parameters f and d, which are positive integers. An initial configuration consists of a pair (G, F), where G is a finite, simple, and undirected graph, and F is a set of f vertices of G. The vertices in F are considered *burned* and are called the *fire sources*. The game proceeds in rounds. In each round, first at most d vertices of G are *defended*, and then all vertices of G that are not yet burned or defended and have a burned neighbor are burned. Once a vertex is burned or defended, it remains so for the rest of the game. The game ends with the first round, in which no further vertex is burned. All vertices of G that are not burned at the end of the game are *saved*. The objective of the player is to save as many vertices of G as possible. Let $s(G, F, d)$ denote this maximum number, that is, a strategy for the game is optimal if it saves $s(G, F, d)$ vertices.

The firefighter game is even hard for $f = d = 1$ and trees of maximum degree 3 [6] or cubic graphs [12]. Approximation algorithms for trees were considered in [10,2–4,11]. An interesting notion in this context is the *surviving rate of a graph G* [7] defined as

$$\rho(G, f, d) = \frac{1}{\binom{n(G)}{f}} \sum_{F \in \binom{V(G)}{f}} \frac{s(G, F, d)}{n(G)},$$

where $n(G)$ denotes the order of G and $\binom{V(G)}{f}$ denotes the set of all subsets of cardinality f of the vertex set $V(G)$ of G. The surviving rate is the average proportion of the vertices of G that can be saved in the firefighter game, where the average is taken over all sets of f fire sources. Equivalently, it is the expected fraction of vertices that can be saved when the fire

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starts at a random set of f vertices. Proofs of lower bounds for the surviving rate often rely on the analysis of very simple strategies for the firefighter game; hence these bounds are essentially average case performance guarantees for these simple strategies. While the firefighter game is hard for trees, Cai et al. [1] gave an elegant argument for the surprising fact that $\rho(T,1,1) = 1 - O\left(\frac{\log n(T)}{n(T)}\right)$ for every tree T, that is, asymptotically almost all vertices of a tree can be saved on average. The surviving rate has been studied in detail for grids [5,8], planar graphs [13,17], and graphs of bounded average degree [14–16]. For more background, we refer to [7].

In [3] we used the argument of Cai et al. to show that $\rho(T,f,d) = 1 - O\left(\frac{\log n(T)}{n(T)}\right)$ for every tree T provided that either $f \le d$ or T has bounded maximum degree. In the present paper we study the surviving rate of general trees for $f > d$. If

$$c(f,d) = \liminf_{n\to\infty}\Big(\min\{\rho(T,f,d) : T \text{ is a tree of order } n\}\Big),$$

then $\rho(T,f,d) \ge c(f,d) - o(1)$ for every tree T of order n. Using this notation, the result of Cai et al. states $c(1,1) = 1$.

In Section 2 we first consider some simple motivating examples. Then, as our main results, we show that $c(f,1)$ is strictly positive for every f, and that $c(2,1)$ lies between $\frac{4}{9}$ and $\frac{3}{4}$.

2. Results

Throughout this section we assume that f and d are fixed positive integers, that is, all asymptotic statements refer to n tending to infinity and hidden constants depend on f and d.

The simplest tree of large maximum degree is the star, which is easy to defend.

Proposition 2.1. $\rho(K_{1,n-1},f,1) = 1 - \frac{2f}{n} + \frac{f^2+f}{n^2} = 1 - O\left(\frac{1}{n}\right)$ *for* $n > f$.

Proof. For the $\binom{n-1}{f}$ sets F of fire sources that do not contain the center of $K_{1,n-1}$, defending the center yields $s(K_{1,n-1},F,d) = n-f$. For the remaining $\binom{n}{f} - \binom{n-1}{f} = \binom{n-1}{f-1}$ sets F of fire sources that contain the center of $K_{1,n-1}$, defending an endvertex not in F yields $s(K_{1,n-1},F,d) = 1$. This implies $\rho(K_{1,n-1},f,1) = \frac{1}{\binom{n}{f}}\left(\binom{n-1}{f}\left(1-\frac{f}{n}\right) + \binom{n-1}{f-1}\frac{1}{n}\right)$, and the desired result follows. □

For the path P_n of order n, which is a tree of bounded maximum degree, we already know that $1 - O\left(\frac{\log n}{n}\right)$, but this can be improved as follows.

Proposition 2.2. $\rho(P_n,f,1) = 1 - O\left(\frac{1}{n}\right)$.

Proof. We call a set F of fire sources *generic*, if for every vertex u in F, the path P_n contains a subpath $P(u)$ of order $4f+1$ with u as its center such that $P(u)$ contains no other vertex from F. It is easy to see that some optimal strategy for the initial configuration (P_n,F) defends some vertex with a burned neighbor for the first $2f$ rounds and terminates then, which implies that $s(P_n,F,1) = n - f - (1 + \cdots + (2f-1)) = n - 2f^2$. Furthermore, it is easy to see that $s(P_n,F,1) \ge n - 2f^2$ for a non-generic set F. This implies the desired result. □

We proceed to the first example showing that $c(f,d)$ is less than 1 for $f > d$.

Proposition 2.3. *If T is the tree of even order n that arises by adding an edge between the centers of two disjoint stars $K_{1,\frac{n-2}{2}}$, then* $\rho(T,2,1) = \frac{3}{4} - O\left(\frac{1}{n}\right)$.

Proof. Let u and v denote the two centers of the stars that form T. For the $2\binom{\frac{n-2}{2}}{2}$ sets F of fire sources that contain either two endvertices adjacent to u or two endvertices adjacent to v, defending u or v respectively yields $s(T,F,1) = n-2$. For the $\left(\frac{n-2}{2}\right)^2$ sets F of fire sources that contain one endvertex adjacent to u and one endvertex adjacent to v, it is optimal to first defend u and then defend an unburned neighbor of v, which yields $s(T,F,1) = \frac{n-2}{2}$. Since only $O(n)$ of the $\Omega(n^2)$ sets of fire sources contain u or v, the desired result follows. □

Proposition 2.3 implies that $c(2,1) \le \frac{3}{4}$. In fact, we believe that the trees considered in Proposition 2.3 represent the worst situation and pose the following.

Conjecture 2.4. $c(2,1) = \frac{3}{4}$.

We proceed to our two main results.

For a rooted tree T with root r and a vertex u of T, let $n_{(T,r)}(u)$ denote the number of descendants of u in T plus one, that is, $n_{(T,r)}(u)$ is the order of the subtree of T that contains u and all descendants of u. Clearly, $n_{(T,r)}(r) = n(T)$ and $n_{(T,r)}(u) = 1$ for a leaf u. It is well known that, if u is a vertex of T that is at maximum distance from r subject to the condition $n_{(T,r)}(u) > \alpha n(T)$