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The chromatic discrepancy of graphs

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ABSTRACT

For a proper vertex coloring c of a graph G, let $\varphi_c(G)$ denote the maximum, over all induced subgraphs H of G, the difference between the chromatic number $\chi(H)$ and the number of colors used by c to color H. We define the **chromatic discrepancy** of a graph G, denoted by $\varphi(G)$, to be the minimum $\varphi_c(G)$, over all proper colorings c of G. If H is restricted to only connected induced subgraphs, we denote the corresponding parameter by $\hat{\varphi}(G)$. These parameters are aimed at studying graph colorings that use as few colors as possible in a graph and all its induced subgraphs. We study the parameters $\varphi(G)$ and $\hat{\varphi}(G)$ and obtain bounds on them. We obtain general bounds, as well as bounds for certain special classes of graphs including random graphs. We provide structural characterizations of graphs with $\varphi(G) = 0$ and graphs with $\hat{\varphi}(G) = 0$. We also show that computing these parameters is NP-hard.

1. Introduction

Consider a proper vertex coloring of a graph G, i.e., no two adjacent vertices get the same color. Let $\chi(G)$ denote the *chromatic number* of G. It is natural to insist that every induced subgraph of G also is colored with as few colors as possible. In particular, for a coloring c of G, let $\varphi_c(G)$ denote the maximum of, over all induced subgraphs H of G, the difference between $\chi(H)$ and the number of colors used by c to color H. We define $\varphi(G)$, called the *chromatic discrepancy* of G, as the minimum $\varphi_c(G)$ over all proper colorings c of G. We obtain several bounds on $\varphi(G)$ and study its relation to other graph parameters.

We consider only finite and simple undirected graphs. We follow basic definitions and notations in [15]. For a graph G, we use $V(G)$, $E(G)$ and $|G|$ to denote the set of vertices of G, the set of edges of G and the number of vertices in G, respectively. Given a graph G and an induced subgraph H of G, we use $G\backslash H$ to denote the induced subgraph of G on the vertices $V(G)-V(H)$. A star graph is a graph isomorphic to $K_{1,n-1}$. The chromatic discrepancy parameters discussed are formally defined below.

Definition 1.1. Let $\mathcal{H}$ be the set of all induced subgraphs and C be the set of all proper colorings of G. We define $\varphi_c(G)$, for $c \in C$, as $\max_{H\in\mathcal{H}}(|c(H)| - \chi(H))$, where $c(H)$ is the set of colors used by c on H. The *chromatic discrepancy* of G, denoted by $\varphi(G)$, is defined as $\min_{c\in C} \varphi_c(G)$. When $\mathcal{H}$ is restricted to the set of all connected induced subgraphs of G, then the corresponding parameter is denoted by $\hat{\varphi}(G)$.

Note that if G is a disconnected graph then $\hat{\varphi}(G)$ is the maximum of $\hat{\varphi}(G_i)$ where G_is are the connected components of G. It is clear from the definition that $\varphi(G) \geq \hat{\varphi}(G)$. It is also clear from the definition that both φ and $\hat{\varphi}$ are monotonically non-decreasing, i.e., $\varphi(H) \leq \varphi(G)$ and $\hat{\varphi}(H) \leq \hat{\varphi}(G)$ for any induced subgraph H of G. Observe that if G is a complete graph,

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then $\varphi(G) = \hat{\varphi}(G) = 0$ and if G is an odd cycle of length more than 7, then[1] $\varphi(G) = 2$ and $\hat{\varphi}(G) = 1$. The smallest graph G where φ and $\hat{\varphi}$ differ is $K_2 \cup K_1$, i.e., the disjoint union of K_2 and K_1. For this graph, $\varphi(G) = 1$ and $\hat{\varphi}(G) = 0$.

The notion of *local chromatic number* $\psi(G)$ of a graph G was introduced by Erdös et al. in [2] and is defined as the minimum over all proper colorings of G, the maximum number of colors present in the closed neighborhoods of vertices in G. It was shown in [2] that there are graphs with local chromatic number 3 and chromatic number greater than k for any positive integer k. In Section 4, we obtain a lower bound on chromatic discrepancy of triangle-free graphs using local chromatic number. As the local chromatic number is bounded from below by the *fractional chromatic number* (see [6]), we obtain a lower bound on chromatic discrepancy using fractional chromatic number as well. Another related notion is *perfect coloring* of a graph defined in [13]. A proper coloring of a graph G is known as perfect coloring if every connected induced subgraph H of G uses exactly $\omega(H)$ number of colors, where $\omega(H)$ is the clique number of H. We prove in Section 5 that the graphs which admit such a coloring are exactly the graphs for which $\hat{\varphi}(G) = 0$. We remark that our parameter φ differs from the *hypergraph (combinatorial) discrepancy* [8] which measures the minimum over all two-colorings of a hypergraph, the maximum imbalance between the cardinality of the color classes of vertices in the hyperedges of the hypergraph.

Finding a coloring that achieves chromatic discrepancy has potential applications in channel allocation problems in wireless networks [12]. The wireless network can be represented as a graph G whose vertices correspond to the various network devices and the edges correspond to communication links between pairs of devices. The goal is to assign channels to communication links without causing any radio interference. A conflict graph G_C of G has vertices corresponding to the edges in G and an edge is present between two vertices in G_C if assigning the same channel to the corresponding communication links can lead to a radio interference. Allocating channels to the communication links is solved using a vertex coloring of G_C, where the set of colors correspond to the set of channels used in the network. Each channel uses a separate frequency band from the available communication band. By reducing the total number of channels (colors), each channel can be assigned a larger bandwidth for faster communication. In addition, reducing the number of different channels (colors) used in subgraphs corresponding to various geographic regions can potentially have two benefits: (a) communication links in these regions can use larger bandwidth and (b) more channels (colors) are available for channel allocation in other co-located wireless networks in these regions.

2. Our results

We give upper bounds on $\varphi(G)$ in terms of chromatic number $\chi(G)$ and independence number $\alpha(G)$. We show that for a graph on n vertices, $\varphi(G)$ is at most $\min\{n/3, \chi(G)(1 - 1/\alpha(G)), n - \chi(G)\}$. This upper bound is tight for complete graphs and graphs of the form $K_t \cup K_{2t}$. We show that for any graph G, $\varphi(G)$ is lower bounded by $(\chi(G) - \omega(G))/2$, where $\omega(G)$ is the clique number. As a consequence, $\varphi(G) \geq \chi(G)/2 - 1$ for triangle free graphs.

For Mycielski graphs M_k of order k, we show that $\varphi(M_k) = \hat{\varphi}(M_k) = \chi(M_k) - 2 = k - 2$. Hence, for Mycielski graphs, the above upper bound is tight up to additive constant and the above lower bound is tight up to multiplicative constant. We also show that for triangle free graphs, $\varphi(G) \geq \hat{\varphi}(G) \geq \psi(G) - 2$ where $\psi(G)$ is the local chromatic number of G. We obtain a lower bound on the chromatic discrepancy parameters for random graphs under the $G(n, p)$ model. We show that for $2\log(n)/n < p < 1/\log^2(n)$ and for a constant $C > 0$, a.a.s., $\varphi(G) \geq \hat{\varphi}(G) \geq \chi(G)(1 - C/\log(np))$.

We provide structural characterization for graphs with $\hat{\varphi}(G) = 0$ and graphs with $\varphi(G) = 0$. We show that graphs with $\hat{\varphi}(G) = 0$ are exactly the class of paw-free perfect graphs (see Fig. 1 for paw graph) and graphs with $\varphi(G) = 0$ are exactly the class of perfect graphs which are complete multipartite. In general, $\varphi(G)$ and $\hat{\varphi}(G)$ of perfect graphs need not be bounded. We show that there are perfect graphs with arbitrarily large $\varphi(G)$ and similarly there are perfect graphs with arbitrarily large $\hat{\varphi}(G)$.

We also obtain bound on the gap between $\varphi(G)$ and $\hat{\varphi}(G)$. We show that $\varphi(G) \leq \hat{\varphi}(G) + \alpha(G) - 1$ for any connected graph G. It is natural to ask whether there exists an optimal coloring of G that achieves $\varphi(G)$ or $\hat{\varphi}(G)$. We show a class of graphs for which $\varphi_c(G)$ and $\hat{\varphi}_c(G)$ obtained by any optimal coloring c is arbitrarily large compared to $\varphi(G)$ and $\hat{\varphi}(G)$. From the computational aspect we show that computing $\varphi(G)$ and $\hat{\varphi}(G)$ are NP-hard.

3. Upper bounds

In this section, we derive upper bounds for the chromatic discrepancy parameters $\varphi(G)$ and $\hat{\varphi}(G)$ in terms of other graph parameters.

In an optimal coloring c of a graph G, $\varphi_c(G)$ is maximized if there is an independent set on $\chi(G)$ vertices of distinct colors. Similarly, $\hat{\varphi}_c(G)$ is maximized if there is a connected induced 2-chromatic subgraph containing vertices of all the colors. Thus we have the following proposition.

Proposition 3.1. *For any graph G, $\varphi(G) \leq \chi(G) - 1$ and for any graph G with at least one edge, $\hat{\varphi}(G) \leq \chi(G) - 2$.*

[1] Let G be an odd cycle $u_0u_1 \dots u_{n-1}u_0$ of length at least 9. Consider any proper coloring c of G. Without loss of generality, we may assume that u_0, u_1 and u_2 are assigned 3 distinct colors. Consider two vertices $v, w \in S = \{u_4, u_5, \dots, u_{n-2}\}$, such that $(v, w) \notin E(G)$ and $c(v) \neq c(w)$. These two vertices exist since $|S| \geq 4$. Let $x \in \{u_0, u_1, u_2\}$ such that $c(x) \neq c(v)$ and $c(x) \neq c(w)$. Then $H = \{x, v, w\}$ is an independent set in G in which each vertex has a different color. For this independent set H, we have $\varphi_c(H) = 2$.