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Efficiently decomposing, recognizing and triangulating hole-free graphs without diamonds

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ABSTRACT

A graph is hole- and diamond-free (HD-free) if none of its induced subgraphs is isomorphic to a chordless cycle of length at least five or to a diamond. Using the clique separator approach and the simple structure of atoms of HD-free graphs, we show how to recognize HD-free graphs in time $\mathcal{O}(n^2)$. One of the main tools is Lexicographic Breadth-First Search (LexBFS); we give two new properties of LexBFS which are essential for reaching the time bound and which hold for any graph. Moreover, we find minimal triangulations of HD-free graphs in time $\mathcal{O}(n^2)$, introducing efficient algorithms for the minimal triangulation of matched co-bipartite graphs and chordal bipartite graphs.

1. Introduction

Cycle properties of graphs and their algorithmic aspects play a fundamental role in combinatorial optimization, discrete mathematics and computer science. Chordal graphs, weakly chordal graphs and perfect graphs are characterized in terms of cycle properties; these classes are of fundamental importance for algorithmic graph theory and various applications. For $k \geq 4$, let C_k denote the chordless cycle with k vertices; a *hole* is a chordless cycle with at least five vertices. For a set $\mathcal{F}$ of graphs, a graph is $\mathcal{F}$*-free* if none of its induced subgraphs is in $\mathcal{F}$. Thus, a graph G is chordal if and only if G is C_k-free for any $k \geq 4$, and G is weakly chordal if and only if G and $\overline{G}$ are C_k-free for any $k \geq 5$, i.e., G and $\overline{G}$ are hole-free. A graph is *chordal bipartite* if it is bipartite and weakly chordal. Equivalently, B is chordal bipartite if and only if it is hole-free and triangle-free.

A *diamond* is a 4-clique minus one edge. Let P_3 denote the chordless path with three vertices and two edges. Obviously, a graph $G = (V, E)$ is diamond-free if for all $v \in V$, the neighborhood of v is P_3-free. Equivalently, a graph is diamond-free if and only if for all of its edges, their common neighborhood is a clique and for all of its non-edges, their common neighborhood is a stable set. Various papers investigate graph classes defined by cycle conditions and additionally being diamond-free; thus, diamond-free chordal graphs are the well-known block graphs — see [11] for various characterizations

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and the importance of block graphs. More recent work studies even-hole- and diamond-free graphs (forbidding also C_4) [25,38], C_4- and diamond-free graphs [14] and a generalization of hole- and diamond-free graphs [10].

In this paper, we focus on the class of hole- and diamond-free graphs (*HD-free graphs* for short) which generalizes in a natural way the important class of chordal bipartite graphs. In particular, we

- describe the clique separator decomposition of HD-free graphs,
- recognize HD-free graphs in time $\mathcal{O}(n^2)$, and
- give an efficient algorithm for minimal triangulation of these graphs.

An important amount of work has been done on *clique separator decomposition* introduced by Tarjan [37] and Whitesides [39]: A *clique separator* (or *clique cutset*) is a clique whose removal increases the number of components of the graph.

Decomposition by clique separators consists in repeatedly copying a clique separator into the components it defines; the final set of subgraphs obtained, which are then devoid of clique separators, are called *atoms*. Clique separator decomposition is hole- and C_4-preserving (see [6] for a comprehensive survey on this decomposition).

A straightforward approach for decomposing a graph into atoms takes $\mathcal{O}(nm)$ time [34]. Note that testing for a hole takes $\mathcal{O}(m^2)$ time [31], and testing for a diamond takes $\mathcal{O}(nm)$ time, so that a brute-force approach for recognizing HD-free graphs would take $\mathcal{O}(m^2)$ time.

Using the approach of [37], structural properties of atoms in some graph classes were recently used to give efficient algorithms for solving various problems such as Maximum Weight Independent Set on these classes (see e.g. [10]). In this paper, we investigate the structure of atoms of HD-free graphs. This atomic structure, however, does not characterize HD-free graphs since diamond is decomposable into two atoms which are complete subgraphs (but is not HD-free).

Another problem related to the presence of chordless cycles is *Minimal Triangulation*, which consists in adding an inclusion-minimal set of edges to obtain a chordal graph. This problem has also given rise to many recent papers. The time bound for this problem, originally $\mathcal{O}(nm)$ [34], was recently lowered to $\mathcal{O}(n^{2.69})$ time [26] and even $\mathcal{O}(n^{\alpha}\log n) = o(n^{2.376})$ [20]. A special issue of Discrete Mathematics [2] is published on the subject, containing a survey on minimal triangulation [19].

The $\mathcal{O}(n^2)$ time bound for clique separator decomposition in this special case is an interesting result in its own right, as there are few other known graph classes where clique separator decomposition can be done faster than in the general case: AT-free claw-free graphs can be decomposed into their atoms in linear time by first computing a minimal triangulation into a proper interval graph, which is linear [30].

One of our main tools for recognizing HD-free graphs efficiently is Lexicographic Breadth-First Search (LexBFS) which was originally designed to determine a perfect elimination ordering on a chordal graph [34]; it is a linear-time breadth-first search which numbers the vertices from n to 1 and chooses at each step a vertex whose label (which is the list of its numbered neighbors) is lexicographically highest.

We show two new properties for LexBFS, which both hold for general graphs:

1. The vertex which is numbered as 1 by LexBFS belongs to no clique separator.
2. For a LexBFS ordering α of a graph G and a clique separator S of G, we show that for any component C of $G(V-S)$, the sub-ordering β of α on $V-C$ is a LexBFS ordering of $G(V-C)$.

Another aspect of our work deals with minimal triangulation of a graph: In the general case, computing a minimal triangulation is a mandatory preprocessing step to ensure an efficient time bound for clique separator decomposition [37]. Here, we will do the exact opposite: We use the atoms to ensure a good time bound for minimal triangulation.

We show how to compute a minimal triangulation of an HD-free atom in $\mathcal{O}(n^2)$ time. For matched co-bipartite graphs, we define a standard minimal triangulation: Computing each added edge will cost constant time. Triangulation of chordal bipartite graphs was investigated by Kloks and Kratsch [24] in the context of computing the treewidth of chordal bipartite graphs, but [24] did not propose an efficient minimal triangulation algorithm for this class; we introduce an $\mathcal{O}(n^2)$ minimal triangulation process for chordal bipartite graphs using the associated Γ-free matrix. Again, few graph classes are known to have a better bound than $\mathcal{O}(nm)$ for this problem: co-comparability graphs and AT-free claw-free graphs have a linear-time triangulation algorithm [30], as well as claw-free graphs with bounded diameter [8], and co-bipartite graphs can be triangulated in the number of edges between one clique and the other [7].

Our results contribute to illustrate how clique separator decomposition can be useful for developing more efficient algorithms; another recent paper applies this to claw-free graphs [8].

The paper is organized as follows: In Section 2, we give the necessary graph notions. In Section 3, we collect some LexBFS tools. In Section 4, we describe the structure of HD-free atoms. In Section 5, we define and investigate diamond-free *multi-matched* graphs which are an important special case of HD-free graphs and whose recognition in time $\mathcal{O}(n^2)$ is the most challenging part of the paper; this is done in Section 6. In Section 7, we give the $\mathcal{O}(n^2)$ time recognition of HD-free graphs. Finally, in Section 8, we address the problem of computing a minimal triangulation for HD-free atoms in $\mathcal{O}(n^2)$ time, before concluding in Section 9.

2. Preliminaries

Basics. The reader is referred to [15], [11] for classical graph definitions and results not given here. Throughout the paper, we consider finite undirected graphs $G = (V, E)$. Let $n = |V|$ and $m = |E|$. The *complement graph* of G is $\overline{G} = (V, E')$ with