Discrete Applied Mathematics 184 (2015) 122–129

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Domination, coloring and stability in P_5-reducible graphs

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ARTICLE INFO

Article history:
Received 13 June 2012
Received in revised form 9 October 2014
Accepted 11 October 2014
Available online 11 November 2014

Keywords:
P_5-free
Domination
Coloring
Stability
Algorithm

ABSTRACT

A graph G is P_5-*reducible* if every vertex of G lies in at most one induced P_5 (path on five vertices). We show that a number of interesting results concerning P_5-free graphs can be extended to P_5-reducible graphs, namely: the existence of a dominating clique or P_3, the fact that k-colorability can be decided in polynomial time (for fixed k), and the fact that a maximum stable set can be found in polynomial time in the class of k-colorable P_5-reducible graphs (for fixed k).

1. Introduction

Given a family of graphs $\mathcal{F}$, a graph G is said to be $\mathcal{F}$-*free* if G has no induced subgraph that is isomorphic to a member of $\mathcal{F}$. When $\mathcal{F}$ contains only one graph F, we say that G is F-*free*. For integer $n \geq 1$, let P_n denote the path on n vertices. A graph G is called P_5-*reducible* if every vertex of G lies on at most one induced P_5. A graph G is called P_5-*sparse* if every subset of six vertices of G induces a subgraph that contains at most one induced P_5. Clearly, every P_5-free graph is P_5-reducible, and every P_5-reducible graph is P_5-sparse. Our purpose here is to extend to the class of P_5-reducible graphs (possibly also to P_5-sparse graphs) some interesting results concerning P_5-free graphs.

In a graph G, a subset D of $V(G)$ is *dominating* if every vertex in $V(G) \setminus D$ has a neighbor in D. Bacsó and Tuza [2], and independently Cozzens and Kelleher [4], proved that every connected P_5-free graph admits a dominating clique or a dominating P_3. We will show that this result still holds for the class of P_5-reducible graphs (indeed for a superclass of that class), and that a more general result holds for P_5-sparse graphs.

For integer k, a k-*coloring* of the vertices of a graph G is a mapping $c : V \to \{1, \ldots, k\}$ such that any two adjacent vertices u and v satisfy $c(u) \neq c(v)$. A graph G is called k-*colorable* if it admits a k-coloring. The chromatic number $\chi(G)$ of a graph G is the smallest integer k such that G admits a k-coloring. Computing $\chi(G)$ is NP-hard; moreover, deciding if a graph admits a k-coloring is NP-complete for every fixed $k \geq 3$ [6], and even in some restricted classes of graphs (planar graphs [6], see also [5]; triangle-free graphs [12], see also [15]; line-graphs [10]; etc.). It is NP-hard to compute the chromatic number of a P_5-graph [11]. In contrast, Hoàng et al. [9] proved that, for every fixed k, one can decide in polynomial time whether a P_5-free graph is k-colorable. We will show that this result can be extended to P_5-reducible graphs.

In a graph G, a *stable set* (also called independent set) is any subset of pairwise non-adjacent vertices. The MAXIMUM STABLE SET PROBLEM (henceforth MSS) is the problem of finding a stable set of maximum size. In the weighted version of this

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http://dx.doi.org/10.1016/j.dam.2014.10.013

problem, we are given a weight for each vertex of G, and the weight of any subset of vertices is defined as the total weight of its elements. The MAXIMUM WEIGHTED STABLE SET PROBLEM (MWSS) is then the problem of finding a stable set of maximum weight. MSS (and consequently MWSS) is NP-hard in general, even under strong restrictions [6,17]. On the other hand, the complexity of MSS in the class of P_5-free graphs is an open problem. This has attracted the attention of many researchers and there are many results dealing with MSS in special subclasses of P_5-free graphs; see [7] for a survey with many references and [13] for a recent result. In [14], it was shown that for fixed k there exists a polynomial-time algorithm that solves MSS (or the weighted version MWSS) in the class of k-colorable P_5-free graphs. We will show here (see Theorem 6.2) that this result can be extended to the class of k-colorable P_5-reducible graphs.

For standard, undefined terms we refer to [3]. For any vertex v in a graph G, we let $N(v)$ denote the set of neighbors of v, and let $M(v) = V(G) \setminus (\{v\} \cup N(v))$. Given a subset A of $V(G)$ and a vertex v in $V(G) \setminus A$, we say that v is *complete* to A if v is adjacent to every vertex of A. For any subset X of $V(G)$, we let $G[X]$ denote the subgraph of G induced by X. The complementary graph of G is denoted by $\overline{G}$.

2. Forbidden induced subgraphs

P_5-**sparse graphs** A simple exhaustive search shows that there are twelve graphs $F_1, \dots, F_{12}$ on six vertices that contain at least two induced P_5's. Each F_i has vertices a, b, c, d, e with edges ab, bc, cd, de (i.e., these five vertices induce a P_5) plus a sixth vertex u, where the neighborhood U_i of u in graph F_i is as follows: $U_1 = \{a\}$; $U_2 = \{b\}$; $U_3 = \{a, c\}$; $U_4 = \{b, d\}$; $U_5 = \{a, e\}$; $U_6 = \{a, c, e\}$; $U_7 = \{a, b\}$; $U_8 = \{a, d\}$; $U_9 = \{a, b, c\}$; $U_{10} = \{b, c, d\}$; $U_{11} = \{a, b, e\}$; and $U_{12} = \{b, c, e\}$. Let $\mathcal{F}^s = \{F_1, \dots, F_{12}\}$. Thus a graph is P_5-sparse if and only if it is $\mathcal{F}^s$-free, and testing if a graph on n vertices is P_5-sparse can be done in time $O(n^6)$ by checking the subgraphs induced by all subsets on six vertices.

P_5-**reducible graphs** Any graph that is not P_5-reducible contains two intersecting induced P_5's, and the union of these two induced P_5's has at most nine vertices. It follows that the family $\mathcal{F}^r$ of minimally non-P_5-reducible graphs is a finite family of graphs with at most nine (and at least six) vertices. A graph is P_5-reducible if and only if it is $\mathcal{F}^r$-free. In consequence, testing if a graph on n vertices is P_5-reducible can be done in time $O(n^9)$ by checking all induced subgraphs on nine vertices. Exhaustive search shows that $\mathcal{F}^r$ contains 138 graphs. We will not show all these graphs here; only a few of them will be of interest to us.

Let F_{13} be the graph with vertices x, u_i, v_i $(i = 1, 2, 3)$ and edges xu_i, u_iv_i $(i = 1, 2, 3)$. Let F_{14} be the graph obtained from F_{13} by adding one edge u_1u_2. Graphs F_{13} and F_{14} are P_5-sparse but not P_5-reducible; indeed they are minimally not P_5-reducible. Let F_{15} be the graph with eight vertices $a_1, \dots, a_4, b_1, \dots, b_4$ such that $\{a_1, \dots, a_4\}$ induces a 4-cycle and for each $i \in \{1, \dots, 4\}$ the neighborhood of b_i is $\{a_i\}$. Let F_{16} be the graph obtained from F_{15} by adding one edge between two non-adjacent vertices of its 4-cycle. Graphs F_{15} and F_{16} are not P_5-sparse; they contain F_4 and F_{10} respectively. Most of our results on P_5-reducible graphs will actually hold for the class of $\{F_1, \dots, F_{16}\}$-free graphs, oftentimes even for a superclass of that class.

3. Bipartite graphs

It is of interest to know the structure of bipartite P_5-sparse graphs and bipartite P_5-reducible graphs as we will use these results in the following sections.

The members of $\mathcal{F}^s$ that are bipartite are $F_1, \dots, F_6$; so a bipartite graph is P_5-sparse if and only if it is $\{F_1, \dots, F_6\}$-free. This statement can be strengthened as follows. For integers k and ℓ with $k \geq 2$ and $\ell \geq 0$, call (k, ℓ)-*squid* any bipartite graph with vertex-set $\{x_0, x_1, \dots, x_k\} \cup \{y_1, \dots, y_{k+\ell}\}$ and edge-set $\{x_0y_i \mid i = 1, \dots, k + \ell\} \cup \{x_jy_j \mid j = 1, \dots, k\}$. We call *squid* any (k, ℓ)-squid for any $k \geq 2$ and $\ell \geq 0$. Vertex x_0 is the *center* of the squid. Note that a P_5 is a $(2, 0)$-squid.

Theorem 3.1. *For a bipartite graph G, the following properties are equivalent:*

(i) *G is P_5-sparse;*
(ii) *G is $\{F_1, \dots, F_6\}$-free;*
(iii) *Each component of G is either a P_5-free graph or a squid.*

Proof. We have (i) $\Rightarrow$ (ii) because each of $F_1, \dots, F_6$ is a 6-vertex graph that contains at least two induced P_5's. Moreover, it is a routine matter to check that every squid is P_5-sparse, and consequently we have (iii) $\Rightarrow$ (i). Now let us prove the implication (ii) $\Rightarrow$ (iii). So consider any bipartite $\{F_1, \dots, F_6\}$-free graph G. We may assume that G is connected, for otherwise it suffices to prove the statement for each component of G. If G is P_5-free, then (iii) holds, so let us assume that G contains an induced P_5. Since a P_5 is a squid, we can consider the largest squid S in G. Let S have vertex-set $\{x_0, x_1, \dots, x_k\} \cup \{y_1, \dots, y_{k+\ell}\}$ and edge-set $\{x_0y_i \mid i = 1, \dots, k + \ell\} \cup \{x_jy_j \mid j = 1, \dots, k\}$ for some $k \geq 2$ and $\ell \geq 0$.

We claim that $G = S$. Suppose the contrary. Since G is connected, there is a vertex u of $G \setminus S$ that has a neighbor in S. If u is adjacent to y_i with $i \in \{1, \dots, k\}$, say $i = 1$, then $\{u, x_1, y_1, x_0, y_2, x_2\}$ induces an F_2 or F_4 (depending on the adjacency between u and y_2), a contradiction. Thus u has no neighbor in $\{y_1, \dots, y_k\}$. If u is adjacent to x_i with $i \in \{1, \dots, k\}$, say $i = 1$, then $\{u, x_1, y_1, x_0, y_2, x_2\}$ induces an F_1, F_3, F_5 or F_6 (depending on the adjacency between u and $\{x_0, x_2\}$), a contradiction. Thus u has no neighbor in $\{x_1, \dots, x_k\}$. If u is adjacent to x_0, then $V(S) \cup \{u\}$ induces a $(k, \ell + 1)$-squid, which contradicts the maximality of S. Thus u is not adjacent to x_0. So it must be that $\ell \geq 1$ and u has a neighbor y_i in $\{y_{k+1}, \dots, y_{k+\ell}\}$, say $i = k+1$.