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Pebble exchange on graphs

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ABSTRACT

Let G and H be graphs with the same number of vertices. We introduce a graph puzzle (G, H) in which G is a *board graph* and the set of vertices of H is the set of pebbles. A *configuration* of H on G is defined as a bijection from the set of vertices of G to that of H. A *move* of pebbles is defined as exchanging two pebbles which are adjacent on both G and H. For a pair of configurations f and g, we say that g is *equivalent* to f if f can be transformed into g by a sequence of finite moves. If G is a 4×4 grid graph and H is a star, then the puzzle (G, H) corresponds to the well-known 15-puzzle. A puzzle (G, H) is called *feasible* if all the configurations of H on G are mutually equivalent. In this paper, we study the feasibility of the puzzle under various conditions. Among other results, for the case where one of the two graphs G and H is a cycle, a necessary and sufficient condition for the feasibility of the puzzle (G, H) is shown.

1. Introduction

A graph is finite and undirected with no multiple edge or loop. For a graph G, we denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. Let G and H be a pair of graphs with n vertices. Let us introduce a graph puzzle (G, H) in which G is a board graph and the vertices of H are pebbles. H is called a *pebble graph*. A configuration of H on G is defined as a bijection from $V(G)$ to $V(H)$. Given a configuration f, it is considered that a vertex $x \in V(G)$ of the board G is occupied by a pebble $y = f(x) \in V(H)$. A move is defined as exchanging two pebbles $y_1 = f(x_1)$ and $y_2 = f(x_2)$, if $x_1x_2 \in E(G)$ and $y_1y_2 \in E(H)$. Then, the resultant configuration g is a bijection from $V(G)$ to $V(H)$ such that $g(x_1) = y_2, g(x_2) = y_1$ and $g(x) = f(x)$ for any $x \in V(G) \setminus \{x_1, x_2\}$. Namely, a graph H represents the exchangeability of the pebbles.

Let us define the *puzzle graph* $\mathrm{puz}(G, H)$ such that $V(\mathrm{puz}(G, H))$ is the set of all the configurations of H on G, denoted by $\mathcal{F}(G, H)$, and $E(\mathrm{puz}(G, H)) = \{(f, g) : f, g \in \mathcal{F}(G, H), f$ can be transformed into g by some move$\}$. Note that $\mathrm{puz}(G, H)$ is isomorphic to $\mathrm{puz}(H, G)$ by the symmetry of the definition of a move.

We say that (G, H) is *transitive* if for any configuration $f \in \mathcal{F}(G, H)$ and for any vertex $x \in V(G)$, a pebble $f(x)$ can be shifted to any other vertex of G by a sequence of finite moves. For a graph G, let $c(G)$ be the number of connected components of G. We say that (G, H) is feasible if $c(\mathrm{puz}(G, H)) = 1$.

Now, we note the background of the pebble exchange model, which is studied in this paper. In robotics, the pebble motion model has been widely studied as a mathematical model, in which multiple objects can move individually on an underlying workspace. In particular, in the case where the workspace is represented by a graph, a movable object can move from the current vertex to one of its unoccupied neighbors in each step.

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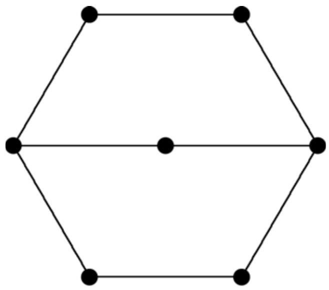

Fig. 1. $\theta(1, 2, 2)$.

In this paper, we introduce the concept of the exchangeability between every pair of objects such that they can swap their positions with each other if and only if they are adjacent to each other both in a board graph and in a pebble graph. This model has a wide range of real world applications. Here we will show two such examples.

Example 1. Let G be a simple graph whose vertex set $V(G)$ is the set of workplaces for robots. Each of the workplaces has a unique electrical outlet and a single robot is working there. Two workplaces u and v in $V(G)$ are adjacent with an edge e of G if there exists a unique passageway from u to v. Each of the passageway is so narrow that at most two robots can pass at the same time. Moreover, there exists a pair of robots such that they have no common method of taking mutual communication and hence the two robots may collide with each other on such a narrow passageway. Let H be a simple graph whose vertex set $V(H)$ is the set of robots working in the workplaces and two robots in $V(H)$ are adjacent if the robots can take mutual communication to avoid their collision. In this case, the rearrangements of the robots $V(H)$ on the workplaces $V(G)$ can be described only by the pebble exchange model on the graph-pair (G, H).

Example 2. Let G be a simple graph whose vertex set is the set of chemical storerooms. In each chemical storeroom, we can store only one type of chemical. Two chemical storerooms u and v in $V(G)$ are adjacent with an edge e of G if there exists a unique passageway from u to v. Again, each of the passageway is so narrow that at most two trucks of chemicals can pass at the same time. There exist several dangerous pairs of chemicals such that, for each of the pairs, a near miss of the two chemicals can cause serious chemical reaction with the possibility of explosion. Let H be a simple graph whose vertex set $V(H)$ is the set of chemicals stored in the chemical storerooms $V(G)$ and two chemicals of $V(H)$ are adjacent if the pair is safe (i.e. no chemical reaction occurs). Now the investigation of rearrangements of the chemicals $V(H)$ in the chemical storerooms $V(G)$ leads us again to treat the pebble exchange model on the graph-pair (G, H).

In the pebble exchange model, the number of pebbles is the same as the number of vertices of a board graph. On the other hand, in an ordinary pebble motion model, the number of pebbles is less than the number of vertices of an underlying graph, because there should be a set of unoccupied vertices as a free space. Hence, the two models may seem quite different at first glance. However, in the pebble exchange model, if a pebble is adjacent to all other pebbles in a pebble graph, it can freely move over the board graph, and so it is regarded to represent the absence of a pebble. Hence, the pebble exchange model is, in a sense, a generalization of the pebble motion model. From this viewpoint, we will restate previously known results on the pebble motion model in terms of our definition.

Suppose that G is a 4×4 grid graph and H is a star $K_{1,15}$. As the center of $K_{1,15}$ is considered as the absence of a pebble, the puzzle (G, H) corresponds to the well-known 15-puzzle [1,6,10].

Wilson studied the problem for the case G is an arbitrary graph and $H = K_{1,n-1}$ by using permutation groups associated with the puzzle [11]. For positive integers a_1, a_2, a_3, we define $\theta(a_1, a_2, a_3)$-graph such that (1) there exists a pair of vertices u and v of degree 3, and (2) u and v are linked by three disjoint paths containing a_1, a_2 and a_3 inner vertices, respectively. For $n \geq 3$, the puzzle $(G, K_{1,n-1})$ is transitive if and only if G is 2-connected.

Theorem A (*Wilson [11]*). *Let $n \geq 3$. Let G be a graph with n vertices. Suppose that G is 2-connected and G is not a cycle. Let $c = c(\mathrm{puz}(G, K_{1,n-1}))$.*

(1) *If G is a bipartite graph, then $c = 2$.*
(2) *If G is not a bipartite graph except $\theta(1, 2, 2)$, then $c = 1$.*
(3) *If G is $\theta(1, 2, 2)$, then $c = 6$.* □

The only exceptional graph in Theorem A is $\theta(1, 2, 2)$. (See Fig. 1.) The puzzle $(\theta(1, 2, 2), K_{1,6})$ is related to many other mathematical objects [4].

For two graphs G_1 and G_2, let $G_1 + G_2$ be the *join* of G_1 and G_2, where $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{x_1x_2 : x_1 \in V(G_1), x_2 \in V(G_2)\}$. Kornhauser et al. considered the puzzle having multiple unoccupied positions [7]. In the following, let us show their results in terms of our setting. Let $p \geq 1$ and $q \geq 2$ be integers with $p + q = n$. Corresponding to the puzzles with q unoccupied positions, Theorem A is generalized for the case $H = \overline{K_p} + K_q$