Discrete Applied Mathematics 184 (2015) 167–170

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Equitable total-coloring of subcubic graphs

Hao Gui [a], Weifan Wang [a,*], Yiqiao Wang [b], Zhao Zhang [a]

[a] *Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China*

[b] *School of Management, Beijing University of Chinese Medicine, Beijing 100029, China*

ARTICLE INFO

Article history:
Received 23 October 2013
Received in revised form 14 November 2014
Accepted 17 November 2014
Available online 12 December 2014

Keywords:
Subcubic graph
Total-coloring
Equitable coloring

ABSTRACT

An equitable total-coloring of a graph G is a proper total-coloring such that the number of vertices and edges in any two color classes differ by at most one. Let $\chi''(G)$ and Δ denote the total chromatic number and the maximum degree of a graph G, respectively. In 1994, Fu conjectured that for any integer $k \geq \max\{\chi''(G), \Delta + 2\}$, G is equitably total-k-colorable. In this paper, we confirm this conjecture for the case $\Delta = 3$.

1. Introduction

Throughout this paper we consider finite loopless multigraphs. Let $V(G)$, $E(G)$, $\Delta(G)$ (for short, Δ), and $\delta(G)$ denote the vertex set, the edge set, the maximum degree, and the minimum degree of a graph G, respectively. For a vertex $v \in V(G)$, let $N_G(v)$ denote the set of neighbors of v in G, and $d_G(v)$ denote the degree of v in G. A *k-vertex* is a vertex of degree k.

A graph G is *totally k-colorable* if the elements of $V(G) \cup E(G)$ can be colored by k colors such that any two adjacent or incident elements have different colors. The *total chromatic number*, denoted by $\chi''(G)$, of G is the least integer k such that G is totally k-colorable. A total-k-coloring of G is *equitable* if the number of the elements in any two classes differ by at most one. The *equitably total chromatic number* $\chi''_e(G)$ of G is the least integer k such that G has an equitable total-k-coloring. By considering coloring only for $V(G)$, we can define equitable k-coloring and equitable chromatic number $\chi_e(G)$.

In 1973, Meyer [9] introduced the notion of equitable (vertex) coloring of graphs and conjectured that the equitable chromatic number of a connected graph G, which is neither a complete graph nor an odd cycle, is at most Δ. Later Chen, Lih and Wu [2] put forward the following stronger conjecture:

Conjecture 1. *Every connected graph G, different from K_m, C_{2m+1} and $K_{2m+1,2m+1}$ for $m \geq 1$, is equitably Δ-colorable.*

Conjecture 1 was confirmed for trees [1], bipartite graphs [8], graphs with $\Delta = 3$ [2], graphs with $\Delta = 4$ [5], outerplanar graphs [7], planar graphs with $\Delta \geq 9$ [10], etc. The following celebrated result was first obtained by Hajnal and Szemerédi [4], whereas a shorter proof appeared in [6]:

Theorem 1. *Every graph G is equitably k-colorable for all $k \geq \Delta + 1$.*

In 1994, Fu [3] first introduced the concept of the equitable total-coloring of graphs (named as equalized total coloring). After some discussions, He presented the following challenging conjecture:

* Corresponding author.
E-mail address: wwf@zjnu.cn (W. Wang).

http://dx.doi.org/10.1016/j.dam.2014.11.014
0166-218X/

Conjecture 2. *For each $k \geq \max\{\chi''(G), \Delta + 2\}$, G is equitably total-k-colorable.*

In [3], the author confirmed Conjecture 2 for a few special cases such as trees, complete graphs, complete bipartite graphs, complete split graphs, and graphs G with $\Delta \geq |V(G)| - 2$. He also noticed that there exist infinitely many graphs G with $\chi''(G) = \Delta + 1$ which is not equitably total $(\Delta + 1)$-colorable.

Recall that a *cubic graph* is a 3-regular graph. A *subcubic graph* is a subgraph of a cubic graph. Wang [11] proved the following result:

Theorem 2. *Every subcubic multigraph is equitably total-5-colorable.*

In this paper, we continue to consider the equitable total-coloring of subcubic graphs. We will extend Theorem 2 to the statement that for any integer $k \geq 6$, every subcubic multigraph is equitably total-k-colorable. This result and Theorem 2 imply that Conjecture 2 is true for the class of subcubic graphs.

2. Results

Assume that G is a graph, which admits an equitable total-k-coloring ϕ using the color set $B = \{1, 2, \dots, k\}$. Set $T(G) = V(G) \cup E(G)$. Then ϕ can be expressed as $(T_1, T_2, \dots, T_k)$, where T_i denotes the ith color class of ϕ. Thus

$$\left\lfloor \frac{|T(G)|}{k} \right\rfloor \leq |T_i| \leq \left\lceil \frac{|T(G)|}{k} \right\rceil, \quad i = 1, 2, \dots, k.$$

Define

$$B_l(G) = \left\{ i \in B : \ |T_i| = \left\lfloor \frac{|T(G)|}{k} \right\rfloor \right\},$$
$$B_u(G) = \left\{ i \in B : \ |T_i| = \left\lceil \frac{|T(G)|}{k} \right\rceil \right\}.$$

We call $B_l(G)$ and $B_u(G)$ the set of *lower* and *upper colors* of ϕ, respectively. If $|T_1| = |T_2| = \cdots = |T_k|$, then $B_l(G) = B_u(G) = B$. Otherwise, $B = B_l(G) \cup B_u(G)$, $B_l(G) \cap B_u(G) = \emptyset$, and $B_l(G), B_u(G) \neq \emptyset$.

As an easy observation, the following Lemmas 1 and 2 appeared in [11]:

Lemma 1. *Suppose that G_1 and G_2 are two vertex-disjoint graphs. If G_1 and G_2 are equitably total-k-colorable, then so is $G_1 \cup G_2$.*

Lemma 2. *Every graph G with $\Delta \leq 2$ is equitably total-k-colorable for any $k \geq 4$.*

A cycle C of a graph G is called *even* or *odd* if $|V(C)|$ is even or odd. A *chord* of C is an edge joining two nonconsecutive vertices on C.

Lemma 3. *Every cubic multigraph G, distinct from K_4, contains an even cycle with at most one chord.*

Proof. If G contains multi-edges, then the result holds trivially. So assume that G is a simple graph. We first show that G contains an even cycle. Let $P = v_1 v_2 \cdots v_m$ be a longest path in G. Then v_2 is adjacent to v_1, i.e., $v_2 \in N_G(v_1)$. By the choice of P, the other neighbors of v_1, other than v_2, must lie on the path P, say $v_i, v_j \in N_G(v_1)$ with $2 < i < j$. If i or j is even, then an even cycle $v_1 v_2 \cdots v_i v_1$ or $v_1 v_2 \cdots v_j v_1$ is found. Otherwise, both i and j are odd, it follows that $v_1 v_i v_{i+1} \cdots v_j v_1$ is an even cycle.

Next, let $C = x_0 x_1 \cdots x_{n-1} x_0$ be a shortest even cycle in G. We shall prove that C has at most one chord. For $p < q$, we use $C[p, q]$ to denote the subset of vertices on C from x_p to x_q along clockwise direction. Namely, $C[p, q] = \{x_p, x_{p+1}, \dots, x_{q-1}, x_q\}$. Moreover, let $C(p, q) = C[p, q] \setminus \{x_p, x_q\}$. If $n = 4$, then C has at most one chord, for otherwise $G = K_4$, contradicting the assumption on G. Hence $n \geq 6$. We assume further that G contains no 4-cycles.

Suppose that C has two distinct chords $e_1 = x_0 x_i$ and $e_2 = x_j x_k$ with $j < k$, where indices are taken modulo n. If $\{0, i\} \cap \{j, k\} \neq \emptyset$, then it is easy to find a shorter even cycle than C, a contradiction. Thus assume that $0, i, j, k$ are mutually distinct. If e_1 and e_2 are two parallel chords of C, say $i < j < k \leq n-1$, then $x_0 x_i x_{i+1} \cdots x_j x_k x_{k+1} \cdots x_{n-1} x_0$ is an shorter even cycle than C, which is a contradiction. So assume that e_1 and e_2 are two crossing chords of C, say $0 < j < i < k \leq n-1$. Since G contains no 4-cycles, it is easy to conclude that $C(0, j) \cup C(i, k) \neq \emptyset$ and $C(j, i) \cup C(k, 0) \neq \emptyset$. If both $|C(0, j)|$ and $|C(i, k)|$ are of same parity, then $C[0, j] \cup C[i, k]$ forms a shorter even cycle, which is impossible. The similar discussion works for $|C(j, i)|$ and $|C(k, 0)|$. Otherwise, without loss of generality, assume that $|C(0, j)|$ and $|C(j, i)|$ are odd, and $|C(i, k)|$ and $|C(k, 0)|$ are even. Now, $x_0 x_1 \cdots x_j x_k x_{k+1} \cdots x_{n-1} x_0$ is an even cycle whose length is shorter than that of C, a contradiction. This completes the proof of the lemma. □

Theorem 3. *Every subcubic multigraph G is equitably total-6-colorable.*