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Domination versus disjunctive domination in trees

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ABSTRACT

A dominating set in a graph G is a set S of vertices of G such that every vertex not in S is adjacent to a vertex of S. The domination number, $\gamma(G)$, of G is the minimum cardinality of a dominating set of G. A set S of vertices in G is a disjunctive dominating set in G if every vertex not in S is adjacent to a vertex of S or has at least two vertices in S at distance 2 from it in G. The disjunctive domination number, $\gamma_2^d(G)$, of G is the minimum cardinality of a disjunctive dominating set in G. It is known that there exist graphs G that belong to the class of bipartite graphs or claw-free graphs or chordal graphs such that $\gamma(G) > C\gamma_2^d(G)$ for any given constant C. In this paper, we show that if T is a tree, then $\gamma(T) \le 2\gamma_2^d(T) - 1$, and we provide a constructive characterization of the trees achieving equality in this bound.

1. Introduction

A *dominating set* in a graph G is a set S of vertices of G such that every vertex not in S is adjacent to at least one vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. The literature on the subject of domination parameters in graphs up to the year 1997 has been surveyed and detailed in the two books [5,6].

Motivated by the concepts of distance domination and exponential domination (see, for example, [1,2,7]), Goddard, Henning and McPillan [4] introduced and studied the concept of disjunctive domination in a graph. A set S of vertices in a graph G is a *disjunctive dominating set*, abbreviated 2DD-set, in G if every vertex not in S is adjacent to a vertex of S or has at least two vertices in S at distance 2 from it in G. We say a vertex v in G is *disjunctively dominated*, abbreviated 2D-*dominated*, by the set S, if $v \in S$ or if v has a neighbor in S or there exist at least two vertices in S at distance 2 from v in G. The *disjunctive domination number* of G, denoted by $\gamma_2^d(G)$, is the minimum cardinality of a 2DD-set in G. Every dominating set is a 2DD-set, implying the following observation.

Observation 1 ([4]). *For every graph G, we have $\gamma_2^d(G) \le \gamma(G)$.*

As observed in [8], the ratio $\gamma(G)/\gamma_2^d(G)$ can be made arbitrarily large, even when restricted to the class of bipartite graphs or claw-free graphs or chordal graphs.

Observation 2 ([8]). *There exist graphs G that belong to the class of bipartite graphs or claw-free graphs or chordal graphs such that $\gamma(G) > C\gamma_2^d(G)$ for any given constant C.*

Unlike the somewhat negative result of Observation 2, we show that if G belongs to the class of trees, then the ratio $\gamma(G)/\gamma_2^d(G)$ is bounded. More precisely, we prove the following result.

Theorem 3. *For every tree T, we have $\gamma(T)/\gamma_2^d(T) < 2$ and this bound is asymptotically tight.*

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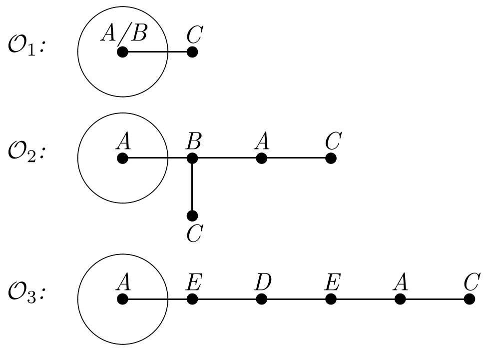

Fig. 1. The three operations $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$.

1.1. *Terminology and notation*

For notation and graph theory terminology not defined herein, we refer the reader to [9]. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ of order $n(G) = |V|$ and edge set $E = E(G)$ of size $m(G) = |E|$, and let v be a vertex in V. We denote the *degree* of v in G by $d_G(v)$. A *path* on n vertices is denoted by P_n. For two vertices u and v in a connected graph G, the *distance* $d_G(u, v)$ between u and v is the length of a shortest (u, v)-path in G. The maximum distance among all pairs of vertices of G is the *diameter* of G, which is denoted by $\text{diam}(G)$. A graph is *nontrivial* if $n(G) \geq 2$.

A *rooted tree* T distinguishes one vertex r called the *root*. For each vertex $v \neq r$ of T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. A *descendant* of v is a vertex $u \neq v$ such that the unique (r, u)-path contains v. Thus, every child of v is a descendant of v. We let $D(v)$ denote the set of descendants of v, and we define $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. A *leaf* of T is a vertex of degree 1, while a *support vertex* of T is a vertex adjacent to a leaf. A *double star* is a tree with exactly two vertices that are not leaves.

The *open neighborhood* of a vertex v is the set $N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is $N_G[v] = \{v\} \cup N_G(v)$. If the graph G is clear from the context, we simply write n, m, $d(v)$, $d(u, v)$, $N(v)$, and $N[v]$ rather than $n(G)$, $m(G)$, $d_G(v)$, $d_G(u, v)$, $N_G(v)$, and $N_G[v]$, respectively.

By a *weak partition* of a set we mean a partition of the set in which some of the subsets may be empty. For our purposes we define a *labeling* of a tree T as a weak partition $S = (S_A, S_B, S_C, S_D, S_E)$ of $V(T)$. We will refer to the pair (T, S) as a *labeled tree*. The *label* or *status* of a vertex v, denoted $\text{sta}(v)$, is the letter $x \in \{A, B, C, D, E\}$ such that $v \in S_x$.

1.2. *The family* $\mathcal{T}$

Let $\mathcal{T}$ be the minimum family of labeled trees that: (i) contains (K_2, S_0^*) where S_0^* is the labeling that assigns to one vertex status A and to the other vertex status C; and (ii) is closed under the three operations $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ that are listed below, which extend the tree T' to a tree T by attaching a tree to the vertex $v \in V(T')$, called the *attacher* of T'. We call the edge that joins v to the vertex of the attached tree, the *attached edge*.

- **Operation** $\mathcal{O}_1$. Let v be a vertex with $\text{sta}(v) \in \{A, B\}$. Add a vertex u_1 and the edge vu_1. Let $\text{sta}(u_1) = C$.
- **Operation** $\mathcal{O}_2$. Let v be a vertex with $\text{sta}(v) = A$. Add a path $u_1u_2u_3u_4$ and the edge vu_2. Let $\text{sta}(u_1) = \text{sta}(u_4) = C$ and $\text{sta}(u_2) = B$, $\text{sta}(u_3) = A$.
- **Operation** $\mathcal{O}_3$. Let v be a vertex with $\text{sta}(v) = A$. Add a path $u_1u_2u_3u_4u_5$ and the edge vu_1. Let $\text{sta}(u_1) = \text{sta}(u_3) = E$, $\text{sta}(u_2) = D$, $\text{sta}(u_4) = A$, and $\text{sta}(u_5) = C$.

The three operations $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ are illustrated in Fig. 1.

2. Main result

Our aim in this paper is twofold: Firstly to present a tight upper bound for the domination number of a tree in terms of its disjunctive domination number, and secondly to provide a constructive characterization of the trees achieving equality in this bound.

The key to our constructive characterization is to find a labeling of the vertices that indicates the role each vertex plays in the sets associated with both parameters. This idea of labeling the vertices is introduced in [3], where trees with equal domination and independent domination numbers as well as trees with equal domination and total domination numbers are characterized.

We shall prove the following result that establishes a tight upper bound for the domination number of a tree in terms of its disjunctive domination number, and provides a constructive characterization of the trees achieving equality in this upper bound. We remark that Theorem 3 is an immediate consequence of Theorem 4.